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TEXT BOOK OF MECHANICS

Designed for Colleges and Technical Schools

By LOUIS A. MARTIN, Jr.

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TEXT-BOOK

OF

MECHANICS

BY

LOUIS A. MARTIN, JR.

Professor of Mechanics
Stevens Institute of Technology

VOL. V.

HYDRAULICS

FIRST EDITION
FIRST THOUSAND

NEW YORK
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LOUIS A. MARTIN, Jr.

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PREFACE

In this text-book the fundamental principles underlying hydraulics have been systematically developed. Especial attention has been devoted to the principles underlying the theory and the design of impulse wheels and turbines.

The numerous exercises should serve to fix these principles in the mind of the student and to assist him to a working knowledge of the subject. Answers to all exercises are given.

The method of presentation is the same as that used in the first four volumes of this series. Again we have sought to produce a text-book which will encourage the student to think and not to memorize, to do and not simply to accept something already done for him. At the same time care has been taken to furnish sufficient material in the way of explanation and example so that the student may not become discouraged.

No attempt has been made to include in this text lists of the experimental constants of hydraulics. For these the student is referred to his engineering hand-books, to treatises on hydraulics, and to the engineering journals.

The assistance rendered by my wife, Alwynne B. Martin, both in the preparation of the manuscript and in the reading of the proof is hereby gratefully acknowledged.

L. A. M., Jr.

Castle Point, Hoboken, N. J.
August, 1914

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HYDRAULICS

INTRODUCTION

MECHANICS may conveniently be divided into three parts: mechanics of solids, of liquids, and of gases. Mechanics of solids, when ideal, rigid solids only are considered, forms the simplest part of the subject, and this is the part first studied under the title of theoretical mechanics. When rigidity is no longer assumed the study of the mechanics of solids involves the elastic properties of materials.

The rational, rigorously developed theory of elasticity is so complicated and leads to results so unsatisfactory from the practical standpoint that engineering students are (and all students should be) first taught a simplified theory of elasticity known as mechanics of materials where, whenever the rigorous theory leads too far afield, recourse is had to approximations and reasonable assumptions based upon observation and experiment.

The study of the mechanics of liquids is divided into hydrostatics and hydrokinetics. In the first division are studied the effects of liquids at rest; in the second liquids in motion.

Hydrostatics is simple and gives but little trouble. Hydrokinetics, the rational science of liquids in motion, is developed from the fundamental postulates of mechanics and leads to results which, while very interesting, are from the engineering standpoint not worth the difficulties overcome in their development. In fact the solution of most of the problems essential to engineers cannot be attained by the rigorous methods of hydrokinetics. Here again recourse must be had to approximations, assumptions, and the empirical results of experiments to assist in the solution of the problems. The science so developed is called **hydraulics.** Although hydraulics means the flow of water in pipes the term is now applied not only to the study of flowing water in general, but even to the study of hydrostatics considered from an engineering standpoint.

As a change in temperature even though small greatly affects the state of a gas, the mechanics of gases is usually studied under the title of thermodynamics. The transfer of heat, the effect of which in the case of gases is of relatively much more importance than in the case of solids and liquids, is there studied in detail.

A liquid may be conceived as composed of particles between which the cohesion is so slight that the particles may change their relative positions on the application of the slightest force. Thus liquids, although preserving their volume under ordinary conditions, do not tend to preserve their shape.

The ideal liquid between whose particles perfect freedom of motion is supposed to exist is called a perfect liquid. In a perfect liquid the particles can transmit force in a direction normal to a surface with which they are in contact, but tangential forces cannot be transmitted to this surface.

Actually all existing liquids are to some degree viscous and the greater their viscosity the more do they depart from the properties assigned to perfect liquids. In the discussion of liquids at rest the viscosity does not affect the solution of the problem, but when motion occurs the viscosity and other frictional resistances must be taken into account.

LIQUIDS AT REST

CHAPTER I

LIQUIDS AT REST

SECTION I

PRESSURE EXERTED BY A LIQUID

LIQUIDS at rest or in very slow motion may be assumed to act as a perfect liquid. The pressures exerted by them are thus always normal to the confining surfaces.

The **pressure** exerted by a liquid is always measured by the force it exerts on a given surface divided by the area of the surface considered. Thus if dA represents the area of the surface, dF the force exerted upon this surface,

then, $p=\frac{dF}{dA}$ is the pressure exerted.

If the pressure p is constant over a surface whose area is A, then the force exerted is

$$F=\int p\,dA=p\int dA=pA,$$

and

$$p=\frac{F}{A}.$$

Pressure is usually expressed in pounds per square inch. Great care must be taken to change the units into pounds per square foot when the pressure is to be substituted in formulas involving other mechanical constants, such as g, in which the unit of length is always the foot.

The Pressure at any Point in a Liquid at Rest is the Same in every Direction.—We are familiar with this fact through its experimental demonstration in the general course in physics. To show that it is a necessary consequence of the absence of tangential forces between the particles of a perfect liquid, imagine a portion of the liquid separated from the surrounding liquid by five imaginary planes as shown in Fig. 1. This portion of the liquid may be made a free body if the action of the surrounding liquid, which is to be imagined as removed, be correctly represented by the forces it exerts upon the element considered. The weight of this element is proportional to the third power of its dimensions, while the forces exerted by the surrounding liquid are proportional to the square of its dimensions; thus in the equations of equilibrium of a differential element the weight would appear as a term of higher order than those due to the pressures, and thus according to the method of the infinitesimal calculus the weight must be neglected.

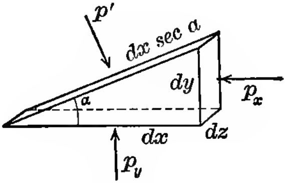


FIG. 1.

Let the pressures on the differential faces be p_x, p_y, and p' as shown. The pressures on the front and rear faces and the forces due to these pressures evidently are not involved in a discussion of the equality of the pressures p_x, p_y, and p'. For equilibrium the sum of the horizontal and the vertical forces parallel to the plane of the paper in Fig. 1 must be separately equal to zero.

Thus

$$(p'\,dx \sec\alpha\,dz)\sin\alpha - p_x\,dy\,dz = 0$$

and

$$-(p'\,dx \sec\alpha\,dz)\cos\alpha + p_y\,dx\,dz = 0,$$

and as

$$\frac{dy}{dx} = \tan\alpha,$$

we have

$$p' = p_x, \quad \text{and} \quad p' = p_y.$$

So that the pressure on any imaginary inclined plane at an angle α equals the pressure on either a vertical or a horizontal plane at the same point in the liquid.

The Increase in Pressure Is Proportional to the Increase in Depth.—The pressure at any point 2 (Fig. 2) will thus equal the pressure at any other point 1 plus a constant multiplied by the vertical distance h.

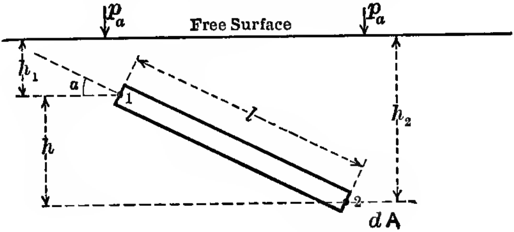


FIG. 2.

The pressure on the free surface of the liquid is usually equal to the atmospheric pressure (normally 14.7 pounds per square inch) denoted by p_a. Let p_1 denote the pressure at the point 1 and p_2 the pressure at the point 2. To obtain an equation connecting these pressures use is again made of the free body method. The most convenient free body here consists of a cylindrical portion of the liquid whose axis joins the points 1 and 2. This cylinder should be imagined of infinitesimal cross-section, dA, so that the pressure distributed over its end sections at 1 and 2 may be constant.

The forces acting upon this free body are the weight of the portion of the liquid considered, $w\,s\,l\,dA$, where

w is the weight of a cubic foot of water,

and s is the specific gravity of the liquid,

and the forces due to the pressures exerted by the surrounding liquid.

As it is now only necessary to take the sum of the forces along the line joining points 1 and 2 in order to obtain the relation between p_1 and p_2, the forces due to the pressures normal to the cylindrical surface of the free body do not appear in the equation for equilibrium.

We have, $$p_1\,dA - p_2\,dA + (wsldA)\sin\alpha = 0,$$

or $$p_2 - p_1 = wsl\sin\alpha = wsh.$$

Thus, the increase in pressure, $p_2 - p_1$, is proportional to the increase in depth, $h = h_2 - h_1$, and moreover equals the increase in depth multiplied by the weight of a cubic foot of the liquid, ws. All this provided the specific gravity, s, of the liquid remains constant with increasing depth. As liquids are practically incompressible the increase in pressure with increasing depth is insufficient to materially affect the specific gravity.

Exercise 1. Show that the pressure is constant at all points in any horizontal plane in a liquid at rest.

Exercise 2. Show that the free surface of a liquid at rest must be a horizontal plane.

Exercise 3. Two liquids which do not mix and whose specific gravities are s_1 and s_2, where $s_1 < s_2$, are superimposed. Show that the common surface of the two liquids must be a horizontal plane.

Exercise 4. Show that the pressure throughout a mass of liquid is constant when the weight of the liquid is neglected.

Exercise 5. A pipe (diameter 12 inches) opens into another pipe whose diameter is three times and whose axis is at right angles to that of the first pipe. A force of 50 pounds is applied to a piston fitted into the smaller pipe. What stress will be produced in a rod $\frac{1}{4}$ inch in diameter joining two pistons fitted to the larger pipe, one on each side of the junction with the smaller pipe? Neglect the weight of the liquid.

Exercise 6. If the diameter of the pump plunger of a hydraulic press is $\frac{1}{2}$ inch what must be the diameter of the ram in order that the mechanical advantage may be 100?

Exercise 7. A force of 50 pounds is applied to a piston whose area is 2.5 square inches. This piston is held in equilibrium by confined water. What pressure does the water exert upon a plane surface of the confining vessel having an area of 10 square inches?

Exercise 8. Water weighs 62.4 pounds per cubic foot. At what depth will water exert a force of 100 pounds upon the side of a box having an area of 2 square inches?

If this box is full of water which is in communication with the external water, what pressure does the water exert upon the side at the same depth?

The Water Barometer.—The pressure exerted by the atmosphere in which we live must always be taken into consideration when the absolute pressure at any point in a liquid is desired. In engineering practice it is often convenient to measure only the pressure in excess of the atmospheric pressure existing at the time of experimentation; this is called the gage pressure. The pressure of the atmosphere as measured with a barometer must be added to this gage pressure in order to obtain the absolute pressure.

A water barometer consists of a tube, of sufficient diameter (not less than ½ inch) so that no correction need be made for capillary action, permanently closed at one end and filled with water. This tube is then temporarily closed at the other end, which is now submerged in a basin of water. If the submerged end of the tube is opened water will flow from the tube into the basin provided the top of the tube is more than 34 feet above the level of the free surface of the water in the basin. When equilibrium has been established the water in the tube will stand about 34 feet above the level of the water in the basin (Fig. 3).

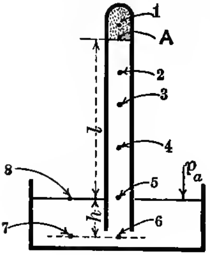


FIG. 3.

That this column of water serves to measure the pressure of the atmosphere is evident from the following discussion. The pressure in the top of the tube is that due to the vapor of water which alone fills this space, as air is supposed to have been excluded. Let this pressure be p_1 at the temperature at which the experiment is made.

If we now consider the pressures at the points A (where the pressure is p_1), 2, 3, 4, it follows from the discussion on page 7 that the pressure increases directly as the depth. At 5 the pressure has become

$$p_5 = p_1 + wl,$$

where w for water is about 62.4 pounds per cubic foot. At 6 the pressure is

$$p_6 = p_1 + wl + wh,$$

and at 7 we have $p_7 = p_6$.

At 8 the pressure is

$$p_8 = p_7 - wh = p_1 + wl.$$

But the only pressure at 8 is the atmospheric pressure which we shall always designate by p_a, thus $p_a = p_1 + wl$.

Exercise 9. At 80° F. the pressure of water vapor is 0.505 pound per square inch and the specific weight of water is 62.22 pounds per cubic foot. Compute the pressure exerted by the atmosphere when a water barometer reads 32 feet at 80° F.

In practice mercury is always used as the liquid in the barometer. Its use has many advantages. The high specific gravity of mercury permits the use of a very much shorter tube and its vapor pressure at ordinary temperatures is so small as to be negligible.

Exercise 10. The atmo (the standard atmosphere) is defined as the pressure due to 760 mm. of mercury at 32° F. The specific gravity of mercury at 32° F. is 13.6 and the specific weight of water at 32° F. is 62.4. Express one atmo in pounds per square inch and in feet of water at 32° F.

Exercise 11. Show that the diameter of a barometer tube cannot affect the height of the column of the liquid provided capillary action can be disregarded.

EXERCISE 12. Assuming the weight of air at the sea-level to be 0.0807 pound per cubic foot and that it remains constant with increasing altitude (which it does not), what would be the height of an incompressible ocean of air producing a pressure of 14.7 pounds per square inch at the sea level?

Measurement of Pressure.—Pressures are usually measured by means of a pressure gage. In experimental work it is often convenient and sometimes more accurate to measure the pressures exerted by confined liquids and gases by means of the height of a column of liquid which the pressure to be measured can sustain.

Thus in Fig. 4 let M represent a water main to which are attached two tubes sufficiently long to prevent the overflow of the water. Such tubes are called **piezometer** (pressure measuring) **tubes.**

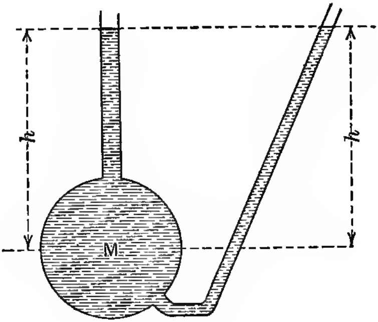


FIG. 4.

The pressure at the free surface in either tube is the atmospheric pressure, p_a. At the center of the main the absolute pressure, p_m, is that due to the atmosphere plus an additional depth of water h,

thus $$p_m = p_a + wh.$$

The gage pressure at M equals $p_m - p_a = wh$.

EXERCISE 13. To what height will water rise in an open piezometer tube fitted to a water main containing water under a pressure of 40 pounds per square inch, gage?

Pressure Head.—Pressures are often expressed in terms of the height to which water would rise in a piezometer tube. This height is called the pressure head. It only indicates the gage pressure. The absolute pressure head would equal the piezometer reading plus the height to which the water would rise in a water barometer.

EXERCISE 14. Show that in the case of water 1 foot of head = .434 pound per square inch and that 1 pound per square inch = 2.30 feet of head.

Gages.—The differential gage illustrated in Fig. 5 is used to measure differences in pressure. To find the difference in the pressures at the centers of the pipes M and N, the head of oil z, the vertical distance, h, between M and N, and the specific gravity of the oil must be known.

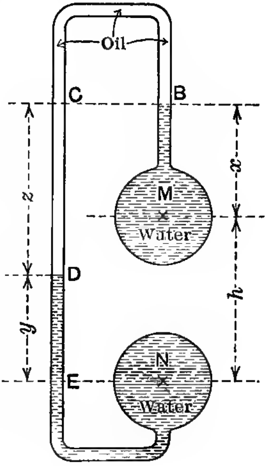


FIG. 5.

Let p_m = the pressure at M,

p_n = the pressure at N,

w = the specific weight of water,

and s = the specific gravity of the oil.

To write an equation involving the pressures p_m and p_n, let us start at M where the pressure is p_m. At B the pressure p_b, is evidently less than the pressure at M by the pressure due to the head x of water.

Thus
$$p_b = p_m - wx.$$

The pressure at C equals the pressure at B. The pressure at D, p_d, is greater than the pressure at C by the pressure due to the head z of oil.

Thus
$$p_d = p_c + wsz = p_m - wx + wsz.$$

At E we have
$$p_e = p_m - wx + wsz + wy.$$

And as the pressure at E equals the pressure at N

$$p_n = p_m - wx + wsz + wy.$$

But Fig. 5 shows that $x + h = y + z$,

so that
$$p_m - p_n = w[z(1 - s) - h].$$

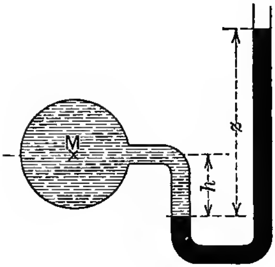


FIG. 6.

EXERCISE 15. Compute the absolute pressure at M (Fig. 6) in terms of z, the reading of the **mercurial gage**, s, the specific gravity of mercury, the distance h, and the pressure of the atmosphere, p_a.

EXERCISE 16. Compute the difference in pressure between M and N in Fig. 5, starting the solution with the pressure at N and working towards the pressure at M.

Exercise 17. Compute the difference in the pressures at M and N, Fig. 7, in terms of x, h, w, and s, the specific gravity of the mercury.

Exercise 18. (*a*) Does the surface of separation at B, Fig. 5, rise or fall with increasing pressure in M and constant pressure in N?

(*b*) Does the top of the left-hand column of mercury, Fig. 7, rise or fall with increasing pressure in M and constant pressure in N?

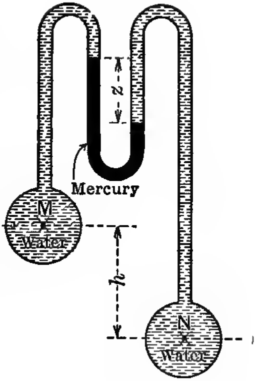


Fig. 7.

Section II

FORCE EXERTED BY A LIQUID

Upon Plane Surfaces.—The force exerted by a liquid upon a plane surface is always normal to the surface and equals the sum of the forces exerted upon the differential elements of the surface. The summation of these differential forces requires the use of the calculus.

As an **example** let it be required to find the force exerted upon the rectangle shown in Fig. 8.

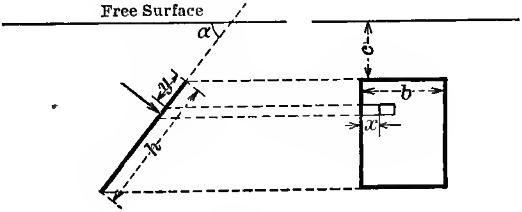


Fig. 8.

Here the **pressure** on any element, in terms of the coordinates shown in the figure, is

$$ws(c+y\sin\alpha)$$

where w is the specific weight of water and s is the specific gravity of the liquid. The **force** on any element, normal to the surface, is

$$ws(c+y\sin\alpha)dxdy.$$

Thus the force exerted on the whole surface is

$$\int\int ws(c+y\sin\alpha)dxdy=ws\int_0^b dx\int_0^h(c+y\sin\alpha)dy$$
$$=wsb\left(ch+\frac{h^2}{2}\sin\alpha\right)=wsbh\left(c+\frac{h}{2}\sin\alpha\right).$$

Note that this result may be interpreted as follows: The total force equals the pressure at the center of area of the surface multiplied by the area of the surface.

Theorem.—The force exerted by a liquid upon any plane surface equals the pressure at the center of area of the surface multiplied by the area of the surface.

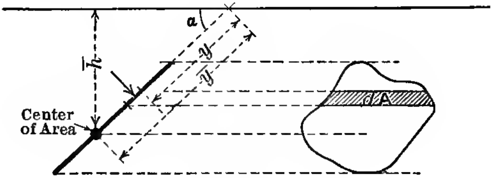


FIG. 9.

The proof of this theorem follows. In Fig. 9 we have as the force exerted by the liquid,

$$F=\int(ws\,y\sin\alpha)dA=ws\sin\alpha\int ydA.$$

But
$$\int y\,dA = \bar{y}A,$$

so that
$$F = ws \sin \alpha\, \bar{y}A = (ws\bar{h})\, A.$$

Exercise 19. A cubical vessel is filled with water. Compute the force exerted by the water upon the bottom and upon a side.

Exercise 20. What force does a liquid (specific gravity, *s*) exert upon each of the surfaces shown in Fig. 10?

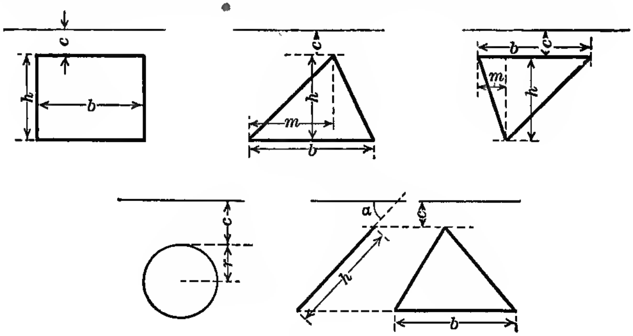


Fig. 10.

When the location of the center of area is unknown the direct application of the calculus as illustrated on page 16 affords a simple solution.

Exercise 21. Same as Ex. 20 for Fig. 11.

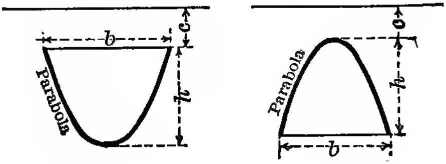


Fig. 11.

Upon Any Surface.—The forces exerted by a liquid upon the elements of a surface which is not plane are not parallel. Their algebraic sum has no mechanical significance.

To find the force exerted upon any surface in a given direction the components of the forces exerted upon the various elements taken in the given direction must be added in order to find the component of the resultant force in that direction.

As an *example* let it be required to find the total vertical force exerted by a liquid (specific gravity s), upon a cylindrical solid (radius, r, altitude, h) immersed so that its highest point lies in, and its axis is inclined at an angle α to, the free surface, Fig. 12.

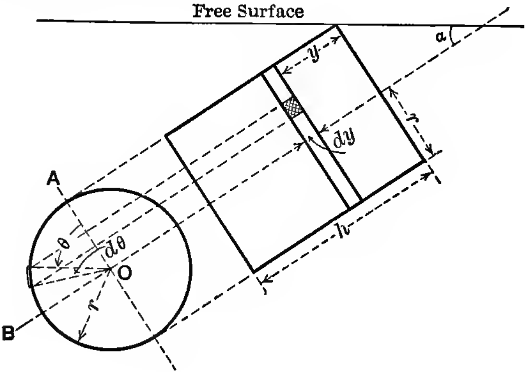


FIG. 12.

In this problem we have three separate surfaces to consider: the two plane surfaces and the cylindrical surface.

Let us first find the force exerted by the liquid upon the cylindrical surface.

The variable coordinates locating an element upon the surface of the cylinder may conveniently be taken as y and θ as shown in Fig. 12. The area of the differential element is then $r d\theta\ dy$, and as its depth below the free surface is $y \sin a + r(1 - \cos \theta) \cos \alpha$, the force exerted upon it is

$$dF = (r d\theta\ dy) ws[y \sin \alpha + r(1 - \cos \theta) \cos \alpha].$$

This force is normal to the cylindrical surface; its direction is therefore the direction of the radius. A component of this force must now be found. Let us find the vertical component. To do this resolve the force dF into two components lying in a plane normal to the axis, one in the direction of OA and the other in the direction of OB. Note that the component parallel to OB has no vertical component. Thus the component parallel to OA is the only one that need be considered. To obtain this component multiply dF by $\cos \theta$. Next resolve this component into two components lying in a vertical plane parallel to the axis of the cylinder. One of these components should be vertical, the other horizontal, so as to have no vertical component. Thus the vertical component of dF is

$$\{dF \cos \theta\} \cos \alpha,$$

and the vertical force exerted upon the cylindrical surface is

$$-\int\int r d\theta\ dy\ ws[y \sin \alpha + r(1 - \cos \theta) \cos \alpha] \cos \theta \cos \alpha$$

$$= -wsr \cos \alpha \Big[\sin \alpha \int_0^h y dy \int_0^{2\pi} \cos \theta\ d\theta +$$

$$r \cos \alpha \int_0^h dy \int_0^{2\pi} \cos \theta\ d\theta - r \cos \alpha \int_0^h dy \int_0^{2\pi} \cos^2 \theta\ d\theta \Big]$$

$$= -wsr \cos \alpha [0 + 0 - \pi h r \cos \alpha] = \pi wshr^2 \cos^2 \alpha.$$

Consider the next vertical force upon the upper plane surface. By means of the theorem on page 16 we obtain the force on this surface. It is

$$wsr \cos \alpha \, (\pi \, r^2).$$

Its vertical component is

$$-\pi wsr^3 \cos \alpha \sin \alpha.$$

Similarly the vertical component of the force on the lower base is

$$+\pi wsr^2(h \sin \alpha + r \cos \alpha) \sin \alpha.$$

The total vertical force exerted upon the cylindrical solid is thus

$$\begin{aligned} &\pi ws \, hr^2 \cos^2 \alpha - \pi ws \, r^3 \cos \alpha \sin \alpha + \pi wsr^2(h \sin \alpha + r \cos \alpha) \sin \alpha \\ &\quad = \pi wsr^2\{h \cos^2 \alpha - r \cos \alpha \sin \alpha + h \sin^2 \alpha + r \cos \alpha \sin \alpha\} \\ &\quad = \pi ws \, r^2 h. \end{aligned}$$

Exercise 22. Find the horizontal component, parallel to the plane of the paper in Fig. 12, of the force exerted on the cylindrical solid.

Exercise 23. Find the horizontal component, normal to the plane of the paper in Fig. 12, of the force exerted on the cylindrical solid.

Exercise 24. Find the resultant force exerted by the liquid upon the cylindrical solid in Fig. 12.

Exercise 25. What force does a liquid filling a spherical shell (internal radius r) exert upon

(*a*) the whole shell?
(*b*) the upper half of the shell?
(*c*) the lower half of the shell?

Exercise 26. A conical shell (altitude, h, radius of the base, r) rests upon its base and is filled with liquid.

Find the force exerted by the liquid

(*a*) upon the base,
(*b*) upon the curved surface.
(*c*) What weight of liquid does it contain?
(*d*) Why does not the force exerted upon the base equal the weight of the liquid contained in the shell?

Special Case.—The horizontal component of the force exerted by a liquid upon a cylindrical surface having equal horizontal elements equals the force exerted upon the vertical projection of this cylindrical surface.

Let Fig. 13 represent the cylindrical surface whose elements are normal to the plane of the paper.

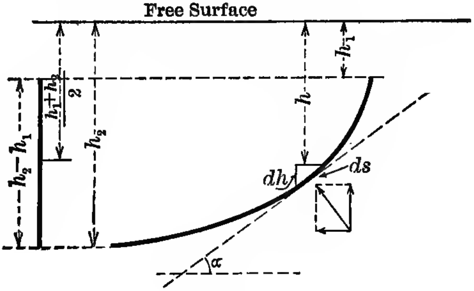


FIG. 13.

Here the pressure on any element is wsh, where h is the variable depth. If the length of the elements of the cylindrical surface is l, the force acting upon an elementary strip of the cylinder is

$$w\,s\,h\,l\,ds,$$

and the horizontal component of this force is

$$w\,s\,h\,l\,ds \sin\alpha.$$

The horizontal force on the whole cylindrical surface is

$$F_H = \int wshl \sin \alpha \, ds,$$

but

$$\sin \alpha \, ds = dh,$$

so that

$$F_H = wsl \int_{h_1}^{h_2} h \, dh = \frac{wsl}{2}(h_2^2 - h_1^2)$$

$$= ws\left(\frac{h_2 + h_1}{2}\right)[l(h_2 - h_1)],$$

which equals the force exerted upon the vertical projection of the cylindrical surface (by the theorem on page 16).

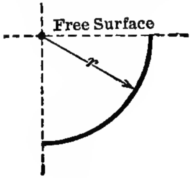


FIG. 14.

EXERCISE 27. Compute the vertical force exerted upon the outside of the cylindrical surface shown in Fig. 14.

EXERCISE 28. Show that the horizontal (vertical) component of the force exerted by a liquid upon a plane surface equals the projected area of the original surface upon a vertical (horizontal) plane parallel to horizontal lines drawn upon the given surface multiplied by the pressure upon the center of area of the original surface.

Centers of Pressure of Plane Surfaces.—It is not sufficient to know the magnitude of the resultant force exerted upon a plane surface by a liquid. The point of application of this resultant force must be known so as to properly allow for its effects. The point in a plane surface through which the line of action of the resultant normal force exerted by a liquid passes is called the center of pressure of the surface.

To Locate the Center of Pressure.—In Fig. 15 let the axis of reference be the intersection of the plane of the surface considered with the free surface of the liquid.

Let $\bar{y}$ be the distance from the center of the area, and y_0 the distance from the center of the pressure to the axis of reference.

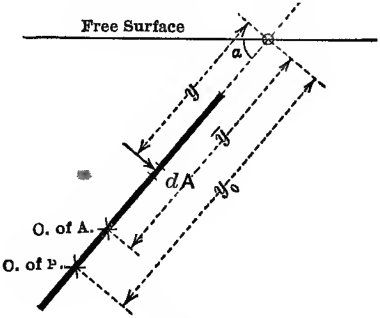


FIG. 15.

The force on any element is

$$ws\, y \sin \alpha\, dA,$$

and its moment about the axis of reference is

$$(ws\, y \sin \alpha\, dA)y.$$

Thus the total moment about the axis of reference due to the pressure of the liquid is

$$\int ws\, y^2 \sin \alpha\, dA.$$

As the moment of a resultant force always equals the sum of the moments of its components and as the resultant force is $\int ws\, y \sin \alpha\, dA$ and as its point of application is the center of pressure we have

$$y_0 \int ws\, y \sin \alpha\, dA = \int ws\, y^2 \sin \alpha\, dA$$

or
$$y_0=\frac{\int y^2 dA}{\int y dA}=\frac{I}{S}, \text{ where}$$

I is the second moment of the area and S is the first moment of the area, both moments to be taken about the axis determined by the intersection of the plane of the surface and the free surface of the liquid.

The expression for y_0 can be simplified by means of the following relations:

$$I=\int y^2 dA=Ak^2 \quad \text{and} \quad k^2=\bar{k}^2+d^2,$$

where k is the radius of gyration of the plane surface considered about the axis of reference; $\bar{k}$ is the radius of gyration of the same surface about an axis parallel to the axis of reference and passing through the center of area of the surface considered; d is the distance between these axes; and A is the area of the surface.

Also
$$S=\int y\, dA=\bar{y}A,$$

where $\bar{y}$ is the distance from the center of area of the surface to the axis of reference, the intersection of the free surface of the liquid with the surface considered;

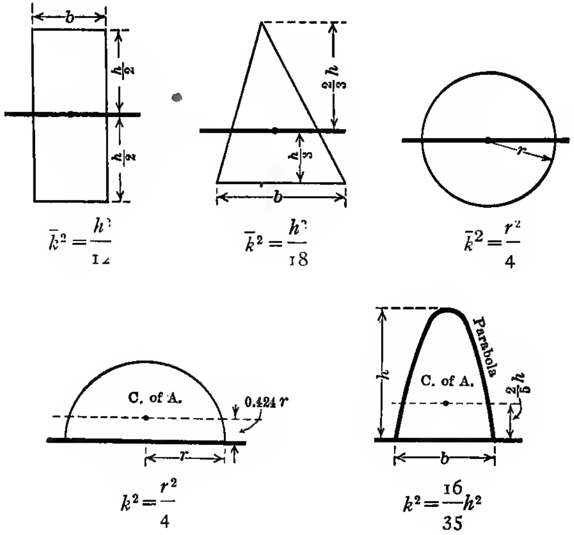
therefore
$$y_0=\frac{I}{S}=\frac{\bar{k}^2+\bar{y}^2}{\bar{y}}.$$

Exercise 29. Prove that $k^2=\bar{k}^2+d^2$.

Exercise 30. Show that the center of pressure is always farther away from the axis of reference than the center of area.

Exercise 31. Show that the center of pressure approaches the center of area as a limit as the depth of the center of area below the free surface increases.

EXERCISE 32. Compute the values of I for the surfaces and and the axes shown in Fig. 16, and thus check the values of k^2 given in the figure.



$\bar{k}^2 = \frac{h^2}{12}$ $\bar{k}^2 = \frac{h^2}{18}$ $\bar{k}^2 = \frac{r^2}{4}$

$k^2 = \frac{r^2}{4}$ $k^2 = \frac{16}{35}h^2$

FIG. 16.

EXERCISE 33. Compute the depth (below the free surface of the liquid) of the centers of pressure of the surfaces shown in Fig. 10.

The lateral location of the center of pressure can in most cases be found by means of the fact that for any surface for which the locus of the mid-points of horizontal lines is a straight line the center of pressure lies upon this line.

That this is so becomes evident when we note that the center of pressure of any horizontal strip of the surface lies on the mid-point of its length. Therefore, provided the mid-points all lie on a straight line the resultant force must also intersect this straight line.

Exercise 34. Locate the center of pressure of the triangle shown in Fig. 17 with reference to the left-hand end of its base.

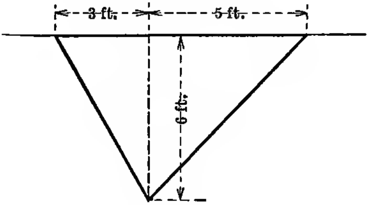


Fig. 17.

The centers of pressure of surfaces built up of simple surfaces, such as illustrated in Fig. 16, may be located by means of the formula

$$y_0=\frac{I}{S}.$$

The values of I and S for the whole surface about the intersection of the surface and the free surface of the liquid as axis must be found.

As an example let us find the center of pressure of the surface shown in Fig. 18.

Divide the surface into two parts, numbered 1 and 2.

The k of 1 about MN equals $\frac{h^2}{12}$, and about AB it equals

$$\frac{h^2}{12}+\left(\frac{h}{2}\right)^2=\frac{h^2}{3}.$$

The k^2 of 2 about CD equals $\frac{r^2}{4}$, about EF (through the center of area of 2) it equals $\frac{r^2}{4}-(.424\,r)^2$, see Fig. 16.

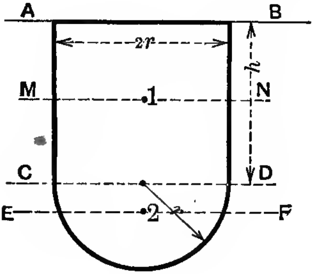


FIG. 18.

Finally, the k^2 of 2 about AB equals

$$\frac{r^2}{4}-(.424\,r)^2+(h+.424\,r)^2=\frac{r^2}{4}+.848\,rh+h^2.$$

The I of the whole surface about AB is thus

$$\left(\frac{h^2}{3}\right)(2rh)+\left(\frac{r^2}{4}+.848\,rh+h^2\right)\left(\frac{\pi r^2}{2}\right).$$

To find S we have the formula

$$S=\bar{y}\Sigma\Delta A=\Sigma y\Delta A,$$

where y represents the ordinate of the center of area ΔA.

The y of the rectangle in Fig. 18 is $\frac{h}{2}$, the area hb.

The y of the semicircle is $h+(.424\,r)$, the area $\frac{\pi r^2}{2}$.

Therefore the S of the whole surface equals

$$\frac{h}{2}\ (hb)+(h+.424\ r)\frac{\pi r^2}{2}.$$

EXERCISE 35. Find the force and the point of application of the force exerted by water upon a vertical surface in the form of an isosceles trapezoid (bases 10 and 4 feet, altitude, 6 feet) the longer base lying in the free surface of the liquid.

SECTION III

APPLICATIONS

Gates and Locks.—A tide gate used to drain a salt marsh is illustrated in Fig. 19. If the gate is 6 feet long the force exerted by the water on the right-hand side is

$$62.5\times6\times12\times6=27{,}000 \text{ pounds.}$$

Its point of application is at

$$y_0=\frac{\frac{(12)^2}{12}+6^2}{6}=8 \text{ feet,}$$

or 4 feet above the top of the sill.

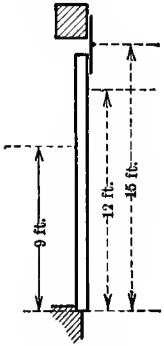


FIG. 19.

On the left-hand side the force exerted by the water equals

$$62.5\times\frac{9}{2}\times9\times6=15{,}200 \text{ pounds,}$$

and this resultant force acts at a point 3 feet above the sill.

EXERCISE 36. Compute the horizontal forces acting upon the hinge and the sill in Fig. 19.

EXERCISE 37. The opening to a drain pipe in a reservoir wall is closed by a circular plate 3 feet in diameter which just covers the pipe and which is hinged to the pipe at its highest point. The plate is inclined at 30° to the vertical and its upper edge is 10 feet below the surface of the water. The plate weighs 300 pounds. Compute the pull in a chain, fastened to the lowest point of the gate and inclined at 45° to the horizontal, necessary to just open the gate.

When *both sides* of a partition are wetted by the same liquid *the resultant force exerted by the liquid upon any portion of the partition completely wetted on both sides passes through the center of area of the portion considered.* Thus in Fig. 19 any portion of the gate below the lower level of water is pushed both towards the right and towards the left by the water with which it is in contact. To prove that the resultant of these two forces passes through the center of area of the portion of the gate considered we may proceed as follows.

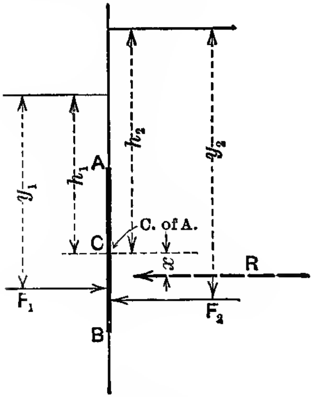


Fig. 20.

In Fig. 20 let AB be the end view of any portion of the gate satisfying the above requirements. Let h_1 and h_2 be the heads upon the center of area of this portion, A of the area AB, and $\bar{k}$ its radius of gyration about a horizontal axis passing through this center of area.

Then the force exerted upon AB on the right is

$$F_2 = ws\, h_2 A,$$

and its point of application is located by means of

$$y_2=\frac{\bar{k}^2+h_2^{\,2}}{h_2}=\frac{\bar{k}^2}{h_2}+h_2.$$

On the left we have

$$F_1=ws\,h_1A,$$

and
$$y_1=\frac{\bar{k}^2}{h_1}+h_1.$$

The resultant force is

$$R=F_2-F_1=wsA(h_2-h_1).$$

Then
$$Rx=F_2(y_2-h_2)-F_1(y_1-h_1),$$

or
$$ws\,A\,(h_2-h_1)x=wsh_2A\left(\frac{\bar{k}^2}{h_2}\right)-wsh_1A\left(\frac{\bar{k}^2}{h_1}\right),$$

therefore,
$$x=0,$$

or, the resultant R passes through C, the center of area of AB.

Exercise 38. Show that the total force exerted on a differential element of the portion AB of the gate considered in Fig. 20 is independent of the position of the element and by means of this fact show that the resultant force passes through the center of area of AB.

Exercise 39. Compute the resultant force exerted upon a portion of the gate (Fig. 19) one foot square whose center of area lies (*a*) 3 ft., (*b*) 6 ft., above the sill.

Exercise 40. A circular opening 6 inches in diameter, center 12 feet below the top of gate Fig. 19 is to be closed by a plate which just fits the opening. Compute the force which the fastenings of the plate will have to withstand.

Exercise 41. Find the force exerted by sea water ($ws=64$) upon the vertical plane end of a pontoon, Fig. 21, floating with a free-board of one foot. Where should this force be applied so as to correctly replace the pressure of the water when the pontoon is regarded as a free rigid body?

In designing the end of the pontoon for strength or for stiffness can the action of the water be replaced by the single force determined above?

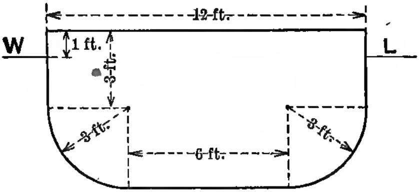


FIG. 21.

Stability of Gravity Dams.—A gravity dam owes its stability principally to its weight. A wall of rectangular section $h' \times b'$ and l' feet long (Fig. 22) retaining water under a head of h feet may as a first approximation be regarded as rigid.

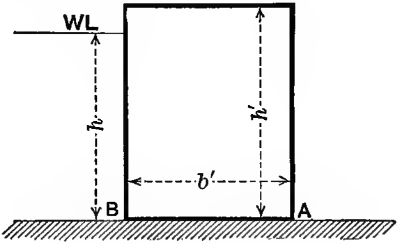


FIG. 22.

If the ground or foundation upon which it rests is also regarded as rigid the dam may be conceived to fail in two ways:

(*a*) by sliding bodily towards the right over the foundation;

(*b*) by tipping (rotating) about its toe A.

The latter assumption is crude and impractical. No concrete or masonry structure has sufficient rigidity to even approximate the condition involved in tipping about A. Moreover, such tipping without crushing of the structure at A with a corresponding shifting of the point of application of the force exerted by the foundations upon the wall would involve infinite resistance to crushing when motion is impending.

Exercise 42. Assuming the specific weight of masonry and water to be w' and w respectively, what must be the width b' to prevent failure:

(*a*) by translation with a coefficient of friction, μ,

(*b*) by rotation about A, Fig. 22?

Exercise 43. Assuming that the slightest tipping has occurred about A, Fig. 22, so that water has penetrated along AB, what must be the least width b' for stability of rotation?

A more reasonable assumption in regard to the normal reaction of the foundation upon the dam, Fig. 22, is to regard this vertical force as distributed along BA instead of concentration at A.

As the law of distribution of the pressure along BA is unknown and unknowable, the simplest and most reasonable assumption is to assume that the pressure variation follows a straight-line law.

As masonry structures should never be relied upon to resist tension, the pressure should never be permitted to diminish below zero. Thus under *the most unfavorable conditions* the pressure exerted by the foundation upon the dam would be zero at B and would reach its greatest value at A, Fig. 22. The pressure is assumed to gradually increase from B to A as shown by the dotted line, Fig. 23. Let the greatest pressure, at A, be p_0, then the pressure at any point at a distance x from B is

$$p=\frac{xp_0}{b'}.$$

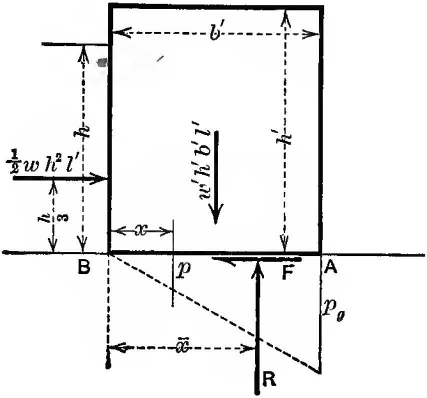


FIG. 23.

The force exerted by the foundation upon any element of the base BA (l' feet long) is

$$pl'dx=\frac{xl'p_0}{b'}dx.$$

The total vertical reaction of the foundation is

$$R=\int_0^{b'}\frac{xl'p_0}{b'}dx=\frac{1}{2}p_0l'b'.$$

If the line of action of R is at a distance $\bar{x}$ from B we have

$$R\bar{x}=\int_0^{b'}\frac{x^2l'p_0}{b'}dx=\frac{l'b'^2p_0}{3}$$

so that $$\bar{x}=\tfrac{2}{3}b'.$$

It should be remembered that the concentrated force R can only be regarded as replacing the distributed force exerted by the foundation when the dam is assumed to be rigid.

Exercise 44. Show that the pressure between the foundation and the dam (Fig. 23) is nowhere less than zero provided the straight-line law of pressure distribution holds and provided

$$b'>h\sqrt{\frac{wh}{w'h'}}.$$

Exercise 45. Show that translation of the dam (Fig. 23) cannot occur if

$$b'>\frac{wh^2}{2\mu w'h'},$$

where μ is the coefficient of friction between foundation and dam.

Exercise 46. Deduce a relation between the dimensions shown in Fig. 24 and the specific weights of masonry and of water (w' and w) under the assumptions that the pressure between the dam and the foundation follows the straight-line law and that the pressure along the inner edge of the base is zero.

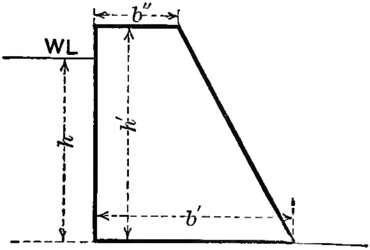


Fig. 24.

In a correctly designed dam the limiting conditions just discussed should never be reached. The pressure between the foundations and the dam should never actually become zero at the inner edge of the base of the dam.

As shown in Fig. 25, assume the pressure at *B* to be p_1 pounds per square foot and at *A* p_2 pounds per square foot; then *ABCD* represents the distributed reaction.

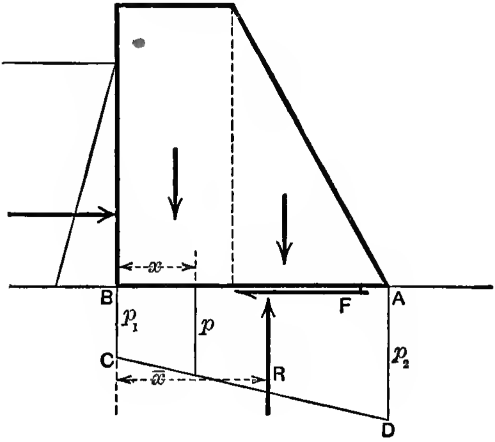


FIG. 25.

If the dam is considered rigid this distributed force may be replaced by a concentrated force. The magnitude of this force may be found by placing the sum of the vertical forces acting upon the dam equal to zero. Its line of action may be located by solving for $\bar{x}$ from the equation obtained by placing the sum of the moments of all forces acting upon the dam about any point (*B* is a convenient point) equal to zero. Note that $\bar{x}$ is not now equal to $\frac{2}{3}(BA)$.

The safety of the dam depends upon the pressures to which the foundations and the material of which the dam is composed are subjected and upon the resistance these materials offer to crushing.

It is thus important to find p_2 (and p_1). Of course the average pressure on BA is $\frac{1}{2}(p_1+p_2)$,

so that, $$\frac{p_1+p_2}{2}\text{ (area of the base } BA) = R \quad . \quad . \quad . \quad (1)$$

but this equation is not alone sufficient to determine either p_1 or p_2.

As R replaces the distributed vertical reaction of the foundation, the moment of R (about any point) must equal the sum of the moments of the individual forces on the elementary areas of BA (about the same point). Thus computing all moments about the point B,

we have, $$R\bar{x} = \int_0^b x(pl\,dx),$$

where b is the width and l is the length of the base BA.

But $$\frac{p-p_1}{x} = \frac{p_2-p_1}{b}$$

If $\frac{p_2-p_1}{b}$ is denoted by c, then $p=cx+p_1$.

Hence, $$R\bar{x} = l\int_0^b (cx^2+p_1x)dx = lb^2\left[\frac{cb}{3}+\frac{p_1}{2}\right],$$

or $$R\bar{x} = \frac{lb^2}{6}(2p_2+p_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By means of Equations (1) and (2) we find that

$$p_2 = \frac{2R}{b^2}(3\bar{x}-b).$$

Exercise 47. A concrete dam of the dimensions shown in Fig. 26 weighs 125 pounds per cubic foot. The water level stands at 9 feet above the foundations.

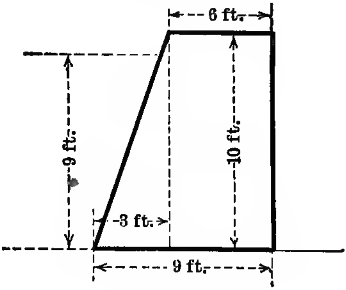


Fig. 26.

(*a*) Compute the greatest and the least pressure exerted upon the foundations, assuming the straight-line law of pressure distribution to hold.

(*b*) Assuming that the slope of the upstream face of the dam remains unchanged but that the width of the rectangular portion is diminished, what width at the base will reduce the least pressure on the foundation to zero?

Section IV

FLOATING BODIES

Buoyant Force.—It has already been shown on page 18 that in the special case there considered the total force exerted by the liquid upon a submerged body equals the weight of the liquid displaced.

A general proof of this principle, known as **the principle of Archimedes,** follows.

Conceive the submerged body divided into vertical cylindrical elements whose horizontal sectional areas, dA, are infinitesimally small. Let the areas of the portions of the surface of the body bounding these cylindrical elements at the top and bottom be dA_1 and dA_2 (Fig. 27), and let their centers of area be h_1 and h_2 feet below the free surface of the liquid. Let θ_1 and θ_2 be the inclinations to the vertical of the normals to dA_1 and dA_2.

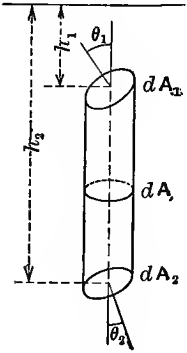


FIG. 27.

Then the pressure on dA_1 is wsh_1, the force on dA_1 is wsh_1dA_1 and its vertical component is

$$wsh_1\, dA_1 \cos\theta_1.$$

The total vertical force exerted by the liquid on the element of the submerged body is

$$dF_v = ws\, h_2\, dA_2 \cos\theta_2 - ws\, h_1\, dA_1 \cos\theta_1.$$

But $\quad dA = dA_2 \cos\theta_2 = dA_1 \cos\theta_1,$

so that $\quad dF_v = ws dA (h_2 - h_1),$

which is the weight of a cylinder of liquid equal in volume to the element of the body considered.

The total vertical force exerted by the liquid upon the submerged body (the buoyant force) is thus equal to the weight of the displaced liquid and its line of action passes through the center of volume (center of gravity) of the submerged body.

EXERCISE 48. Show that the above reasoning applies to a partly submerged (floating) body.

To show that the horizontal force exerted by a liquid upon a submerged body is always zero, conceive the body divided into horizontal slabs of infinitesimal thickness. Fig. 28 shows a top view of such a slab whose depth below the free surface may be assumed to be h feet. The pressure on any element of surface wetted by the liquid will be wsh, irrespective of its inclination, θ, to the horizontal. The horizontal component of the force on any element is

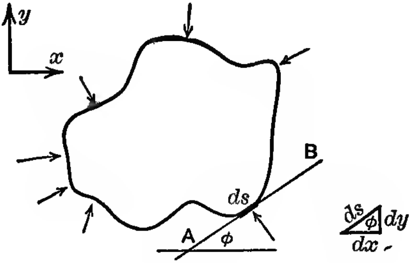


FIG. 28.

$$wsh\,dA\sin\theta,$$

where $dA \sin\theta$ is the projection of dA on a vertical plane tangent to a horizontal section of the body as AB, Fig. 28, so that

$$dA\sin\theta = ds\,dh.$$

Therefore the force acting on the element dA in the direction X is

$$dF_x = (wsh\,dh\,ds)\sin\phi$$

or,

$$F_x = wsh\,dh\int ds\sin\phi,$$

$$F_x = wsh\,dh \int_{y_1}^{y_1} dy = 0.$$

Similarly $\qquad F_y = 0.$

Therefore no resultant horizontal force acts upon any horizontal slab of the submerged body.

Thus: *The resultant force exerted by a liquid upon a submerged (or floating) body always equals the weight of the liquid displaced and acts vertically upward through the center of gravity of the displaced liquid.*

As an application of this principle consider the following problem.

A tank containing water rests upon a platform-scale. The scale registers 300 pounds. An iron casting suspended from a spring balance by means of a fine wire weighs 70 pounds. This casting is now lowered into the water in the tank until it is completely submerged but not in contact with the tank. No water overflows. Under these new conditions what do the spring balance and the platform scale read? (Specific gravity of iron 7.0.)

Fig. 29 (*a*) illustrates the problem. Applying the free body method to this problem we may represent the casting as a free body as in Fig. 29 (*b*). As the casting weighs 7×62.4 pounds per cu.ft., 70 pounds of casting displace $\frac{70}{7 \times 62.4}$ cu.ft. of water. This water weighs $\frac{70}{7 \times 62.4} \times 62.4 = 10$ pounds. So that the total effect of the water upon the casting may be represented by a vertical upward force of 10 pounds as in Fig. 29 (*b*). In addition to this force the weight of the casting, 70 pounds, and the tension in the wire, T pounds, act upon the casting when regarded as a free body. Therefore, $T-70+10=0$ or $T=60$ pounds, is the new reading of the spring balance.

Next represent the tank and the contained water as a free body, Fig. 29 (*c*). As the casting is here assumed removed, its action upon the water must be represented by a suitable force, the reaction of the 10 pounds in Fig. 29 (*b*). Also the weight of the tank and the water, 300 pounds, and the upward force exerted by the platform scales must be shown.

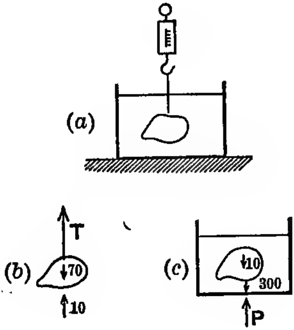


FIG. 29.

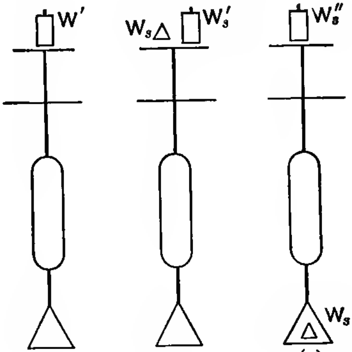
Evidently we have, $P-300-10=0$ or $P=310$ pounds, the new reading of the platform-scales.

EXERCISE 49. Find the force exerted upon the platform scale in the above example by considering the tank, the contained water, the casting, and the suspending wire all together as a free body.

EXERCISE 50. A tank and the contained water weigh 600 pounds. Two cubic feet of cork (specific gravity 0.25) are completely submerged in this water and fastened to the bottom of the tank by a fine wire. What is the tension in this wire?

Nicholson's Hydrometer.—Nicholson's hydrometer consists of a hollow metal float to which are attached two pans as shown in Fig. 30. When in use this instrument is always immersed to the same mark on the upper stem, thus always displacing the same volume of liquid.

It may be used to find the ratio of the specific weight of a given solid, w_s, to the specific weight of a given liquid, w, in which the solid is insoluble. To find this ratio the following weights must be determined by experiment:



(a) (b) (c)

FIG. 30.

W', the weight on the upper pan required to sink the hydrometer to the mark on its stem;

W_s', the weight on the upper pan which together with the specimen to be tested will sink hydrometer to the same mark;

W_s'', the weight on the upper pan necessary to sink the hydrometer to the same mark when the specimen is on the lower pan.

To establish the required ratio let

V = the volume of liquid displaced by the hydrometer;
W = the weight of the hydrometer;
W_s = the weight of the solid specimen;
w_s = the specific weight of the solid specimen;
and w = the specific weight of the liquid.

Then from the equilibrium of the hydrometer under the conditions shown in Fig. 30 (*a*)

we have,
$$W+W'=wV, \quad \ldots\ldots \quad (1)$$

From Fig. 30 (*b*),
$$W+W_s+W_s'=wV, \quad \ldots\ldots \quad (2)$$

and from Fig. 30 (*c*),
$$W+W_s''+W_s=wV+\frac{W_s}{w_s}w. \quad \ldots\ldots \quad (3)$$

We must now eliminate W, W_s, and V from these equations, for these are unknown, and then solve for $\frac{w_s}{w}$.

Subtracting (1) from (2),
$$W_s+W_s'-W'=0. \quad \ldots\ldots \quad (4)$$

Subtracting (2) from (3),
$$W_s''-W_s'=\frac{W_s}{w_s}w; \quad \ldots\ldots \quad (5)$$

from (5)
$$\frac{w_s}{w}=\frac{W_s}{W_s''-W_s'},$$

and by (4)
$$\frac{w_s}{w}=\frac{W'-W_s'}{W_s''-W_s'}.$$

Exercise 51. W_H is the weight of a Nicholson hydrometer. A weight W must be placed in the upper pan to sink this hydrometer to a certain depth in a liquid whose specific weight is w. A weight W' is required to sink it to the same depth in a liquid whose specific weight is w'. Find $\frac{w'}{w}$.

Exercise 52. A common hydrometer floats in a liquid of specific weight w with h inches of its stem exposed and with h' inches exposed in liquid of specific weight w'. The volume of the whole hydrometer is V and the sectional area of its stem is a. Find $\frac{w'}{w}$.

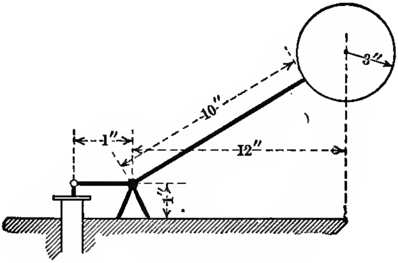


Fig. 31.

Depth of Flotation.—The depth to which the lowest point of a floating body sinks is called the depth of flotation.

Exercise 53. Find the depth of flotation of a right cylinder (specific weight, w'; length, l') floating in a liquid (specific weight, w) provided its axis remains vertical.

Exercise 54. A sphere (radius, 2 feet; specific gravity, 0.4) floats in water; find its depth of flotation. Solve the resulting equation by Newton's method of approximation.

EXERCISE 55. Find the depth of flotation of a right, circular cone (radius, r; altitude, h; specific weight, w) floating in a liquid (specific weight, w') with its base submerged and horizontal.

EXERCISE 56. How high above the floor of the tank, Fig. 31, must the water rise so as to stop the flow through a pipe having a sectional area of $\frac{1}{4}$ sq.in. supplying water under a pressure of 60 pounds per sq.in.? The rods have a sectional area of $\frac{1}{2}$ sq.in. and weigh 0.3 pound per cubic inch. The hollow sphere, diameter 6 inches, weighs one pound.

Stability of Flotation.—Referring to Exercises 53 and 55 it is evident that the position assigned to these floating bodies can be stable only when the specific weight of the bodies and the liquid bear a certain relation to each other. It is this stability of flotation and the positions taken by bodies floating in equilibrium which are now to be investigated.

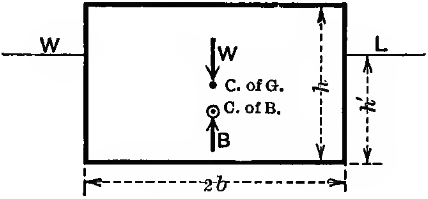


FIG. 32.

As a simple introductory example consider a rectangular parallelepiped floating in the position shown in Fig. 32. The forces acting upon this body are its weight, W, concentrated at its center of gravity and the resultant action of the liquid which can be represented by a vertical force whose line of action passes through the center of gravity of the displaced liquid and whose magnitude equals the weight of the displaced liquid. This force is called the buoyant force and its point of application is called the center of buoyancy.

The forces W and B are evidently in equilibrium when the body floats in the position shown in Fig. 32.

Let us now investigate the conditions when the floating parallelepiped is displaced by an external couple through

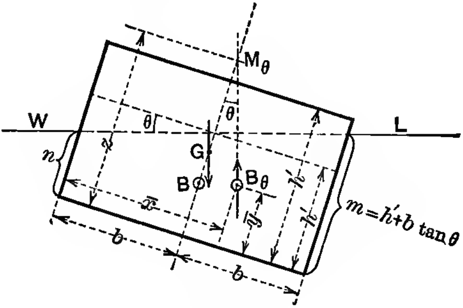


FIG. 33.

an angle θ about an axis perpendicular to the plane of the paper in Fig. 32.

This heeling of the body causes the center of buoyancy to shift from B to B_θ, B_θ being the center of gravity of the displaced liquid which is now a prismatic volume of trapezoidal section. The weight W and the buoyant force now exert a torque, a righting torque, upon the floating body. If an equal and opposite torque is not applied to the body it will after several oscillations return to its original position of equilibrium.

To compute the righting torque acting on the body when displaced as shown in Fig. 33, B_θ must be located.

The center of area of the trapezoidal section of the displaced water is readily found by means of Fig. 34.

Here
$$\bar{x}=\frac{(2bn)(b)+b(m-n)\left(\frac{4b}{3}\right)}{2bn+b(m-n)}$$
$$=\frac{n+2m}{3(m+n)}2b$$
$$=\frac{3h'b+b^2\tan\theta}{3h'},$$

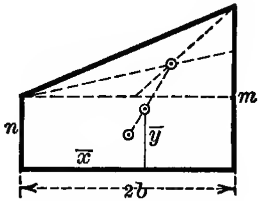


FIG. 34.

and
$$\bar{y}=\frac{(2nb)\left(\frac{n}{2}\right)+b(m-n)\left(n+\frac{m-n}{3}\right)}{2bn+b(m-n)}$$
$$=\frac{3n^2+(m-n)(m+2n)}{3(m+n)}=\frac{m^2+mn+n^2}{3(m+n)}$$
$$=\frac{3h'^2+b^2\tan^2\theta}{6h'}.$$

The arm of the righting couple can now most readily be found by means of the point M_θ, Fig. 33. This point is called a *metacenter* of the body. It is located at the intersection of a vertical line through the center of buoyancy of the displaced body and of the line passing through the center of gravity of the body and the center of buoyancy of the body when it floats in equilibrium under the action of gravitation. The location of this metacenter evidently changes with the displacement of the body.

From Fig. 33,

$$\begin{aligned} z &= \bar{y} + (\bar{x} - b) \cot \theta \\ &= \frac{3h'^2 + b^2 \tan^2 \theta}{6h'} + \left(\frac{3h'b + b^2 \tan^2 \theta}{3h'} - b\right) \cot \theta \\ &= \frac{3h'^2 + b^2(2 + \tan^2 \theta)}{6h'} \\ &= \frac{h'}{2} + \frac{b^2}{6h'}(2 + \tan^2 \theta). \end{aligned}$$

Thus the distance from G to M_θ equals

$$z - \frac{h}{2} = \frac{h' - h}{2} + \frac{b^2}{6h'}(2 + \tan^2 \theta).$$

The righting torque equals

$$W\left(z - \frac{h}{2}\right) \sin \theta,$$

where W represents both the weight of the body and the weight of the displaced water.

In order to determine the stability of a floating body acted on only by its weight and the buoyant force it is not necessary to determine the righting moment for a finite angular displacement of the body. A body is in stable equilibrium when it returns to its initial position after the *slightest displacement.* If a slight displacement leads to a still greater displacement then the equilibrium is unstable. A body is in neutral equilibrium when it is indifferent to slight displacements.

The point M_θ in Fig. 33 approaches a definite position as θ approaches zero as a limit. This limiting position of the point M_θ is called **the metacenter of the body.** From Fig. 33 it is evident that whenever the metacenter of a body lies above its center of gravity the couple called into action by the slightest displacement is a righting couple and under these conditions stable equilibrium exists. If the metacenter lies below the center of gravity the couple acting after the slightest displacement is an upsetting couple and the equilibrium is unstable. Neutral equilibrium exists when the metacenter and the center of gravity coincide.

EXERCISE 57. How high above the base is the metacenter of the body shown in Fig. 32?

Can you determine from the data in the above example whether or not the body is in stable equilibrium?

EXERCISE 58. A plank ($h \times b \times l$, where $l > b > h$) has a specific gravity s, and floats in water. What is the least value for b for which the plank can float with its dimension h vertical?

If $s = 0.4$, what is the least value of b in terms of h?

It would be impractical to locate a metacenter M_θ for a given displacement θ and then by means of it obtain the position of the metacenter by placing $\theta = 0$ as above illustrated for each and every problem that may arise. Instead the formula about to be developed should always be used.

Consider any body symmetrical about a vertical meridian plane, Fig. 35. Let it be displaced through a differential angle $d\theta$, the original water line being WL, the new water line W_1L_1. Then if B and B_1 be the original and the new centers of buoyancy respectively, then M is the metacenter.

The shifting of B to B_1 is due to the shifting of the differential wedge from WOW_1 to LOL_1. The centers of gravity of these wedges lie on the line W_1L_1 (or WL) at g' and g''.

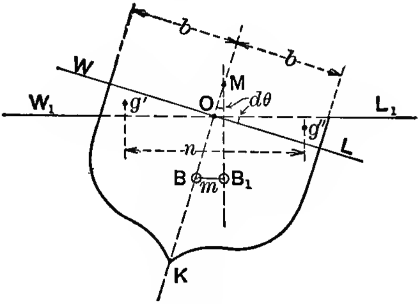


FIG. 35.

To find the position of B_1 let dV = the volume of the wedge, and V = total volume of displacement, then as shown in Fig. 36 we have a weight wV at B of which the weight wdV at g' is a part. The weight of that part of the displaced water $W_1 O L K$ which does not change its position during the displacement of the body is $w(V-dV)$ and it may be conceived as concentrated at B'. During the transfer of wdV from g' to g'', wV is transferred from B to B_1 while $w(V-dV)$ remains stationary at B'.

Therefore
$$(wdV)x = w(V-dV)y$$

or
$$\frac{x+y}{y}=\frac{V}{dV},$$

but from the similar triangles
$$\frac{x+y}{y}=\frac{n}{m},$$

so that
$$\frac{n}{m}=\frac{V}{dV},$$

or
$$n\,dV=m\,V.$$

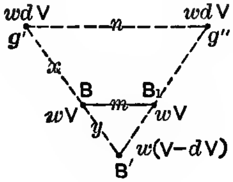


FIG. 36.

Moreover, from Fig. 35, we see that
$$m=\overline{BM}\sin d\theta,$$

that
$$dV=\int\tfrac{1}{2}(b)(bd\theta)dl=\tfrac{1}{2}\int b^2d\theta\,dl,$$

where dl is an element of the length of the body,

and that
$$n=2\{\tfrac{2}{3}b\}=\tfrac{4}{3}b.$$

Therefore
$$n\,dV=m\,V$$

becomes
$$\tfrac{4}{3}b(\tfrac{1}{2}\int b^2\,d\theta\,dl)=\overline{B\,M}\sin d\theta\,V,$$

so that
$$\overline{B\,M}=\frac{\frac{2}{3}\int b^3 d\theta dl}{V \sin d\theta}.$$

As sin $d\theta = d\theta$ and $d\theta$ is a constant, we have

$$\overline{B\,M}=\frac{\frac{2}{3}\int b^3\, dl}{V}.$$

Consider now the water-line section of the floating body, Fig. 37. The second moment of area of this section is

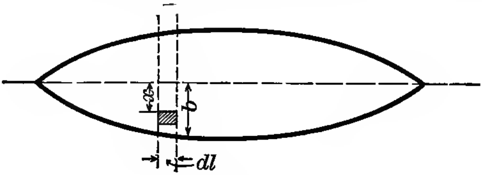


FIG. 37.

$$I=\int_{-b}^{b}\int_{0}^{l} x^2 dx dl = \tfrac{2}{3}\int b^3 dl;$$

therefore
$$\overline{B\,M}=\frac{I}{V}.$$

In words, the distance between the center of buoyancy and the metacenter of a floating body equals the second moment of area of the water-line section about its longitudinal, meridian axis divided by the volume of liquid displaced.

EXERCISE 59. A homogeneous rectangular parallelepiped $l \times b \times b$ where $l > b$ has a specific gravity of 0.2. (*a*) Will it float in stable equilibrium in the position shown in Fig. 38? (*b*)

Between what limits may the specific gravity of a body of this shape and size vary, provided the body is to float in stable equilibrium in the position shown in Fig. 38?

EXERCISE 60. Compute the location of the metacenter of a body of the shape described in Exercise 59 with reference to its center of gravity for a specific gravity of (*a*) 0.1, (*b*) 0.5, (*c*) 0.9.

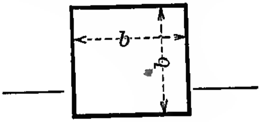


FIG. 38.

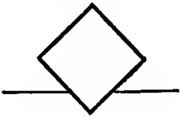


FIG. 39.

EXERCISE 61. For what specific gravities will a body of the shape described in Exercise 59 float in stable equilibrium in the position shown in Fig. 39?

EXERCISE 62. What must be the least specific gravity of a homogeneous cylindrical rod (radius *r*, length *l*) in order that it may float in water with its axis vertical?

LIQUIDS IN MOTION

CHAPTER II

THE FREE SURFACE OF LIQUIDS MOVING WITH ACCELERATION

SECTION V

TRANSLATION

LET AB in Fig. 40 represent the free surface of a liquid in a tank when at rest. Experience teaches that if this tank is set in motion with an acceleration in the direction AB, then the liquid will pile up on the side A. We now seek the nature of the new free surface MN.

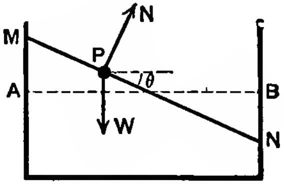


FIG. 40.

Consider any particle of liquid at the free surface such as P, Fig. 40. The forces acting on this particle are the normal force, N, exerted by the surrounding particles, and its weight W. If the liquid and tank move with an acceleration a towards the right, then the resultant of the forces N and W must produce a horizontal acceleration a in the particle.

Thus $$N \cos \theta - W = 0,$$

and $$N \sin \theta = \frac{W}{g} a.$$

Solving for θ we have

$$\tan \theta = \frac{a}{g}.$$

If the acceleration is constant then the angle θ is constant and the free surface becomes a plane inclined at an angle θ and relatively at rest with respect to the tank.

Exercise 63. A rectangular vessel ($nk \times b \times l$) is filled to a depth k with a liquid. If this vessel is moved in the direction of its dimension b with a constant acceleration a, how much liquid overflows?

Exercise 64. Find the center of pressure on the bottom of the tank in Exercise 63 if the acceleration is such that θ equals an arctan $\frac{1}{2}$ and no liquid overflows.

Exercise 65. Find the pressure on the bottom of the tank in Fig. 40 if the water in it is h feet deep and the tank is on an elevator ascending with a downward acceleration of a feet per second per second.

Section VI

ROTATION

A liquid contained in a cylindrical vessel which is whirled about its vertical geometrical axis will sooner or later whirl with the angular velocity of the vessel. This is of course due to the friction which always exists between solid and liquid, and liquid and liquid, particles in relative motion.

Assuming that the cylindrical vessel moves with constant angular velocity and that a condition of relative rest between the vessel and the contained liquid exists, let us find the form of the free surface of the rotating liquid.

In Fig. 41 is shown a particle on the free surface as a free body. The resultant force on the particle must now be such as to produce an acceleration $\omega^2 x$ directed normally towards the axis of rotation. Here

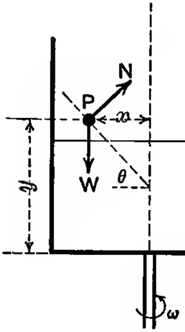


FIG. 41.

ω = the angular velocity of the system;

x = the distance of the particle from the axis of rotation or the radius of its circular path.

If θ is the inclination of a line tangent to a meridian section of the free surface at P (Fig. 41) then we have

$$N \cos \theta - W = 0;$$

$$N \sin \theta = \omega^2 x \left(\frac{W}{g}\right);$$

thus $$\tan\theta=\frac{\omega^2 x}{g}.$$

To determine the equation of the meridian section of the surface of revolution forming the free surface, assume the coordinates of P as in Fig. 41,

then $$\frac{dy}{dx}=\tan\theta,$$

and $$\frac{dy}{dx}=\frac{\omega^2 x}{g}$$

is the differential equation of the curve sought.

Integrating, we have

$$y=\frac{\omega^2}{2g}x^2+c_1,$$

and assuming $y=a$, when $x=0$, we have $c_1=a$,

so that $$y-a=\frac{\omega^2}{2g}x^2.$$

This equation shows that the free surface is a paraboloid of revolution and a represents the depth of liquid at the axis of revolution.

EXERCISE 66. Compute the depth, h, of the parabolic cup Fig. 42.

EXERCISE 67. Show that the lowest point of the free surface when the vessel and the liquid (Fig. 42) are in rotation is as far below the free surface when at rest as the highest point is above this free surface.

EXERCISE 68. If k is the depth of the liquid in the vessel (Fig. 42) when at rest, how fast must it be whirled in order that the liquid may be just on the point of overflowing?

Exercise 69. The vessel in Fig. 42 is initially $\frac{3}{4}$ full of water. Its height is 6 inches; its radius 2 inches.

(*a*) Find the least number of revolutions per minute that this vessel must make if the cylindrical surface is to be wholly wetted.

(*b*) At what speed will the center of the base become dry?

If the top of the cylindrical vessel is closed as in Fig. 43 and the speed is sufficiently high the top will be wetted. Under these conditions we still have

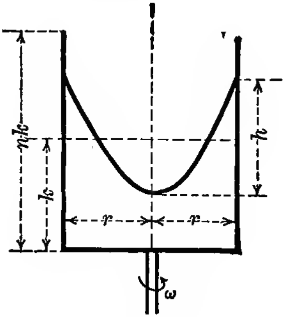


Fig. 42.

$$h' = \frac{\omega^2 r'^2}{2g},$$

but r' is now the radius of the cup and not of the cylinder.

Exercise 70. Show that in Fig. 43 $h' = r\omega\sqrt{\frac{k(n-1)}{g}}$.

Exercise 71. A glass cylinder such as is shown in Fig. 43 can be used as an indicator of speed of rotation. Discuss the nature of the scale which would have to be etched on this cylinder so that revolutions per minute could be read directly.

Exercise 72. A closed cylindrical vessel is full of liquid (specific weight, w) and rotates with constant angular velocity ω about its geometrical axis. Neglecting the weight of this liquid sketch a differential element of the rotating liquid, referred to cylindrical coordinates, as a free body. From this compute the radial pressure at any point in the liquid.

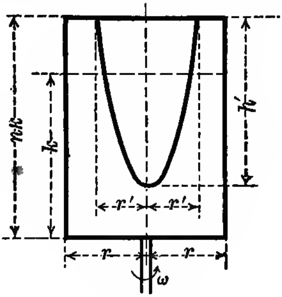


Fig. 43.

CHAPTER III

THE FLOW OF LIQUIDS

SECTION VII

BERNOULLI'S EQUATION AND THE EQUATION OF CONTINUITY

Steady Flow.—Steady flow exists whenever all particles reaching any given point possess the same velocity and are under the same pressure. At any given point the conditions remain unchanged during steady flow. This does not imply that the velocity and the pressure may not vary from point to point in the flowing liquid. During steady flow the particles are assumed to move along definite and fixed paths. These paths are called **stream lines.** Any set of stream lines completely surrounding any portion of flowing liquid may thus be conceived as an imaginary tube separating the portion considered from the rest of the flowing liquid. During steady flow no liquid passes through the walls of this imaginary tube. Thus the normal cross-section of this tube must vary inversely as the velocity of flow for the quantity of liquid passing any section per unit of time must equal the quantity passing any other section per unit of time, for liquids are assumed to be incompressible.

If A_1 and A_2 are the sectional areas of a tube bounded by stream lines at any two points, and v_1 and v_2 are the velocities of flow at these sections, then the flow per unit of time is

$$Q=A_1v_1=A_2v_2.$$

This is known as the **equation of continuity.**

Steady flow is never found in practice. Liquids always flow with eddying motions. Also the velocity of flow at and near a solid boundary is always less than at points in the same section further removed from the bounding surface.

We shall assume that all particles in the same section normal to the flow have the same velocity, that steady flow exists, and that the liquid is frictionless. Any discrepancies between the results computed under these assumptions and the results obtained by experiment must be corrected by numerical multipliers or coefficients determined experimentally.

Bernoulli's Theorem.—The fundamental theorem upon which all calculations involving the flow of liquids are based was demonstrated in 1738 by Daniel Bernoulli. This theorem establishes the relation between the pressures, the velocities, and the positions of the particles of a perfect liquid flowing through an imaginary tube bounded by stream lines.

In Fig. 44 is shown a portion of liquid bounded by stream lines and two normal sections numbered 1 and 2.

Let dA_1 and dA_2 be the areas of the sections at 1 and 2,
h_1 and h_2 their location with reference to any datum plane,
v_1 and v_2 the velocities of flow, and
p_1 and p_2 the pressures in the liquid at 1 and 2.

If dQ is the quantity of liquid forced into the tube across the section 1 in a time dt, then

$$dQ = dA_1(v_1\,dt),$$

and as no liquid flows across the stream lines an equal quantity of liquid must leave the tube at 2.

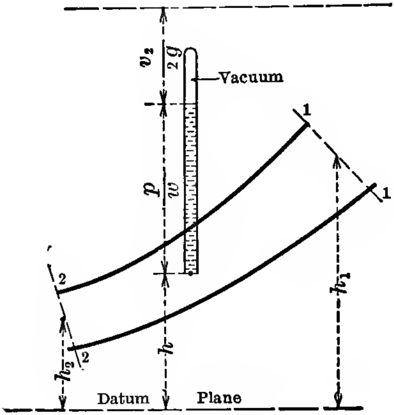


FIG. 44.

Thus $$dQ = dA_1(v_1\,dt) = dA_2(v_2\,dt).$$

The principle of work and energy may now be applied to the portion of the liquid bounded as above described. This liquid considered as a free body is acted upon by the forces due to the pressures exerted by the surrounding liquid and by its weight. The work performed by these forces during the flow which occurs in the time dt must equal the increase in kinetic energy which occurs during this same interval of time.

The work due to the pressure of the surrounding liquid is due wholly to the displacements at sections 1 and 2, for no flow (displacement) occurs across the stream lines. The force acting on section 1 is $p_1\,dA_1$ and the displacement at this section during the time dt is $v_1\,dt$; the work performed is thus,

$$p_1\,dA_1\,v_1\,dt.$$

Similarly at section 2 the work performed is

$$-p_2\,dA_2\,v_2\,dt.$$

The work performed by gravity during the time dt is evidently due to the descent of dQ cubic feet of liquid through a vertical distance of h_1-h_2 feet and equals

$$(wdQ)\;(h_1-h_2).$$

The total work due to all forces is thus

$$p_1\,dA_1\,v_1\,dt-p_2\,dA_2\,v_2\,dt+wdQ(h_1-h_2)$$

and this must equal the gain in kinetic energy

$$\frac{1}{2}\left(\frac{wdQ}{g}\right)v_2^2-\frac{1}{2}\left(\frac{wdQ}{g}\right)v_1^2.$$

Thus, by reason of, $dQ=v_1\,dA_1\,dt=v_2\,dA_2\,dt$

we have $$p_1+\frac{wv_1^2}{2g}+wh_1=p_2+\frac{wv_2^2}{2g}+wh_2.$$

Or we may state that

$$p+\frac{wv^2}{2g}+wh=\text{a constant.}$$

In this form, Bernoulli's equation calls attention to the fact that the total energy per unit volume of a perfect liquid remains constant during steady frictionless flow. Bernoulli's theorem is thus a special case of the general theorem of the conservation of energy.

Bernoulli's theorem is usually remembered in the following, more convenient, form:

$$\frac{p}{w}+\frac{v^2}{2g}+h=\text{const.}$$

Here, $\frac{p}{w}$ is evidently the **pressure head.** It is the height to which the liquid would rise in a piezometer tube whose upper end connects with a perfect vacuum, for p is the absolute pressure of the liquid. $\frac{v^2}{2g}$, by analogy, is called the **velocity head,** and **h** is evidently the **position head,** or the height of the point in the liquid filament above the assumed reference or datum plane.

In Fig. 44 are shown the position head, h, the height of the liquid in the piezometer tube, $\frac{p}{w}$, and to these heads must be added the velocity head, $\frac{v^2}{2g}$, in order to obtain the total head.

EXERCISE 73. In Fig. 44, must the pressure heads at 1 and 2 be equal; must the velocity heads at 1 and 2 be equal; must the total heads at 1 and 2 be equal?

Problems relating to the steady flow of liquids in which the liquids may be assumed to move in stream lines and without frictional losses can be solved by means of: the equation of continuity

$$Av=\text{const.}$$

and Bernoulli's equation

$$\frac{p}{w}+\frac{v^2}{2g}+h=\text{const.}$$

As **an example** let us find the pressure p_3 at 3 in the pipe line shown in Fig. 45. The areas of the sections of the pipe are given as $A_2=2$ sq.ft., $A_3=\frac{1}{2}$ sq.ft., and $A_4=1$ sq.ft.

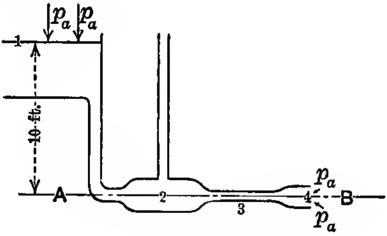


FIG. 45.

Assume the datum plane at AB and let p_a represent the atmospheric pressure. In order to apply Bernoulli's equation two points in the stream must be decided upon and the conditions there existing must be connected by means of the equation. Point 3 should not be selected as one of these points, even though the pressure at 3 is required, for neither the pressure nor the velocity at 3 are known. Note that at 4 the pressure is known to be atmospheric while the velocity is unknown. At 1, the free surface of the reservoir, the pressure is also atmospheric and here the velocity may be assumed to be zero, for the area of the stream section at 1 is very large, whence by reason of the equation of continuity,

$$A_1 v_1 = A_4 v_4,$$

v_1 must approach zero as A_1 approaches infinity, for $A_4 v_4$ must remain finite and not zero.

Thus, applying Bernoulli's equation to points 1 and 4 we have,

$$\frac{p_a}{w} + \frac{0}{2g} + 10 = \frac{p_a}{w} + \frac{v_4^2}{2g} + 0,$$

whence,
$$v_4 = \sqrt{20g} = 25.4 \text{ ft. per sec.}$$

The velocity at 3 can now be found by means of the equation of continuity, thus

$$v_3(\tfrac{1}{2}) = (25.4)\ (1)$$

$$v_3 = 50.8 \text{ ft. per sec.}$$

Now applying Bernoulli's theorem to sections 3 and 4 we have,

$$\frac{p_3}{w} + \frac{(50.8)^2}{2g} + 0 = \frac{p_a}{w} + \frac{(25.4)^2}{2g} + 0,$$

whence,
$$\frac{p_3 - p_a}{w} = -30.1 \text{ feet of head,}$$

or
$$p_3 - p_a = (-30.1)(62.5) = -1880 \text{ pounds per sq.ft.,}$$

$$p_3 = p_a - 13.1 \text{ pounds per sq.in.}$$

The pressure in the pipe at 3 is thus 13.1 pounds per sq.in. below atmospheric pressure. If the atmospheric pressure is 14.7 pounds per sq.in. then the absolute pressure in the pipe is 1.6 pounds per sq.in.

Exercise 74. How high above the datum plane will the water rise in the piezometer tube at 2 (Fig. 45)?

What velocity head exists at 2 (Fig. 45)?

Exercise 75. In Fig. 45 assume $A_4=3$ sq.in., $A_3=1$ sq.in. Compute the discharge at A and the pressure at 3.

Pressures Can Never be Negative.—A negative absolute pressure obtained in the solution of a problem on the flow of liquids means that the liquid is in tension. As liquids can withstand practically no tension a negative absolute pressure is incompatible with steady flow. Under such conditions the liquid column parts and the space becomes partly filled with vapor. Bernoulli's theorem and the law of continuity now no longer apply.

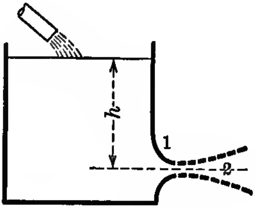


Fig. 46.

Exercise 76. Compute the least area at 3, Fig. 45, consistent with steady flow under the assumptions made in Exercise 75.

Exercise 77. Does the pressure of a liquid increase when it is squeezed through a tube of decreasing section? Support your answer by a discussion based on Bernoulli's theorem.

Exercise 78. An orifice discharges water at a constant head of h feet, Fig. 46. The area of this orifice is A_1. It is proposed to add a flaring pipe through which the water is to discharge.

How large can the area at 2 be made and continuity of flow be maintained?

Exercise 79. (*a*) If the head in Fig. 46 is 10 feet, how much larger than the section at 1 can the section at 2 be made and continuity of flow maintained?

(*b*) How does the discharge through the flaring pipe compare with the discharge through the original orifice?

(*c*) Can the discharge be still further increased by increasing the flare of the pipe?

(*d*) What explanation can you offer for the increased discharge through the flaring pipe?

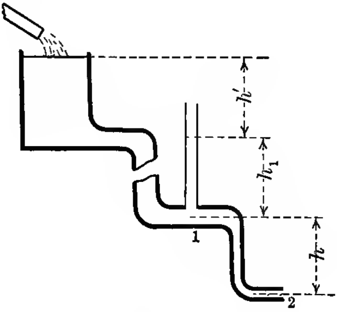


FIG. 47.

Exercise 80. Find the velocity of discharge at 2, Fig. 47, in terms of h, h_1, and the areas A_1 and A_2 at 1 and 2 respectively.

Exercise 81. Show that h' in Fig. 47 equals $\frac{v_1^2}{2g}$, where v_1 is the velocity at 1.

Exercise 82. The tank of a locomotive tender is to be filled from a track trough. With what velocity, relative to the tender, is the water delivered 7 feet above the trough when the tender is moving at 30 miles per hour? Assume that friction and shock cause a loss of 10 feet of head.

Exercise 83. Assuming no loss in Exercise 82, find the least speed of the tender at which water will be delivered to the tank.

The Venturi Water Meter.—The transformation of pressure into velocity during the flow of water through a pipe of decreasing section may be used to measure the rate of flow through the pipe. This is the theory underlying the action of the Venturi meter. As Fig. 48 shows, the diameter of the pipe is diminished to about $\frac{1}{3}$ the diameter of the main pipe in a length of about twice the diameter of the main pipe. The increase in diameter is then gradual, the angle of divergence being about 5°, so that the velocity head may be transformed back into pressure head with as little loss as possible. The pressures in the main at 1 and in the throat at 2 are measured. The quantity of water flowing per second can then be computed in terms of the areas at 1 and 2.

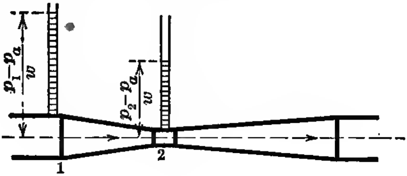


Fig. 48.

In practice the flow so computed must be corrected by the use of a coefficient to be determined by experiment. The coefficient allows for the friction in the converging portion of the pipe.

Exercise 84. Show that the flow through a Venturi meter is

$$Q=nA\sqrt{\frac{2g(p_1-p_2)}{w(n^2-1)}},$$

where p_1 and p_2 are the gage pressures at 1 and 2, Fig. 48, A is the sectional area of the throat of the meter, and n is the ratio of the area of the main to the area of the throat. Frictional losses to be neglected.

What change must be made in the above formula when the pressures are expressed in absolute pressures?

The Energy of Moving Liquids.—The three terms in Bernoulli's equation are called heads. They deal with pressure, velocity, and position. It should, however, be remembered that these terms also represent the energy per pound of moving liquid. That $\frac{p}{w}$ represents the energy per pound of moving liquid due to its pressure, p, becomes evident when we imagine this moving liquid to act upon a piston of area A. The force it would exert is $p\,A$, the displacement of the piston per pound of liquid supplied would be $\frac{1}{wA}$, and the work done by the pressure alone would be

$$(pA)\left(\frac{1}{wA}\right)=\frac{p}{w}.$$

The kinetic energy of one pound of water moving with a velocity v is of course $\frac{1}{2}\left(\frac{1}{g}\right)v^2=\frac{v^2}{2g}$. The potential energy with reference to some datum plane and due to its weight is h.

If these three forms of energy could be wholly extracted from the moving water the available energy per pound of water would be

$$\frac{p}{w}+\frac{v^2}{2g}+h.$$

EXERCISE 85. What power is available at a point in a pipe (6 inches in diameter) at which the gage pressure is 100 pounds per square inch and the velocity of flow is 8 feet per second?

EXERCISE 86. A turbine is supplied with water under a gage pressure of 200 pounds per square inch through a 6-inch pipe. The water leaves the turbine through a 12-inch pipe. A pressure gage attached to this pipe 4 feet below the point at which the first pressure is measured reads 18 pounds per square inch. The discharge pipe runs full and discharges 4 cubic feet per second. Compute the efficiency of the turbine if its brake horse-power is 155.

CHAPTER IV

FLOW THROUGH ORIFICES

SECTION VIII

SMALL ORIFICES

The Standard Orifice.—The velocity with which a liquid leaves an orifice and the quantity of liquid discharged depend not only upon the head at the orifice, upon the size of the orifice, and upon the liquid, but also upon the shape and the nature of the orifice. Orifices used for measuring the discharge of liquids are almost always circular. The shape of the wall bounding an orifice greatly affects both the velocity attained by the liquid and the quantity of liquid discharged. The sharp-edged orifice, Fig. 49 (*a*), allows free contraction of the jet and causes but little loss of energy (and thus velocity) due to friction, for the liquid touches the wall along a line only. If the orifice

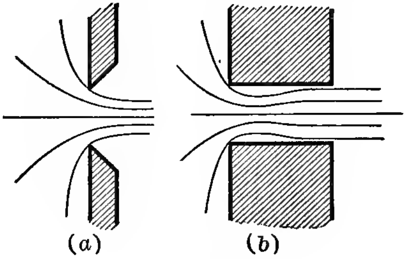


FIG. 49.

consists of a cylindrical hole, Fig. 49 (*b*), in a wall 1.5 or more times as thick as the diameter of the orifice, the jet after contracting on passing the inner edge again expands and discharges full bore. In this case, although the velocity of discharge is less owing to the greater loss due to friction and the production of eddies, the discharge may be greater than the discharge through a sharp-edged orifice under the same conditions, due to the larger cross-section of the issuing jet. If the orifice consists of a cylindrical hole in a thin wall the formation of the jet is the same as in the case of a sharp-edged orifice.

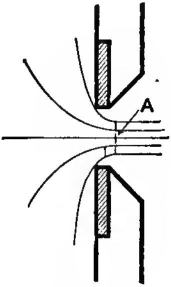


FIG. 50.

It is impossible to compute the losses due to friction and to the production of eddies which occur during the flow through orifices by means of rational mechanics, so that recourse must be had to experimental investigation. Empirical constants, the use of which will be discussed shortly, have been established for various kinds of orifices when discharging under various conditions. These constants can be found in treatises on hydraulics and in the engineering hand-books.

The orifice for which these empirical constants have been most accurately established and which can most easily be reproduced to conform with the conditions involved is called the *standard orifice*. It is a circular, sharp-edged orifice fashioned as shown in Fig. 50. The metal plate containing the orifice proper should be set in the wooden backing and the orifice be placed far enough from the side walls and from the bottom of the tank containing the liquid so that the flow may be interfered with as little as possible and that full contraction of the jet may occur.

The Velocity of Efflux.—Torricelli stated in 1643, as a result of his observations, that the velocity with which a liquid leaves an orifice is equal to the velocity attained by a body falling freely through a height equal to the head of liquid upon the orifice.

Exercise 87. Show that Torricelli's theorem (as stated above) is a special case of Bernoulli's theorem.

The velocity of efflux is equal to

$$v=\sqrt{2gh},$$

where h is the head at the orifice, only under ideal conditions. The viscosity of the liquid and the losses at the orifice cause the actual velocity of efflux to be less than this ideal value.

Coefficient of Velocity.—The actual velocity of efflux may be put equal to

$$v_a=c_v\sqrt{2gh},$$

where c_v is the coefficient of velocity. This coefficient must be determined by experiment. For water flowing through a standard orifice c_v has an average value of 0.98.

The coefficient of velocity is most readily determined by measuring the coordinates of several points on the parabolic trajectory of a jet issuing from an orifice in a vertical plane. By means of these coordinates the actual velocity of efflux may be computed by the methods used in solving problems in projectiles.*

EXERCISE 88. If x and y are the horizontal and vertical coordinates of a point of the jet measured from the orifice (more accurately from the vena contracta) as origin, determine the actual velocity of efflux; also the coefficient of velocity if h represents the head on the orifice (vena contracta).

The Vena Contracta.—The jet issuing from a standard orifice rapidly narrows until it attains its least cross-section at A, Fig. 50. This section is called the vena contracta. The liquid does not attain its full velocity of efflux at the orifice. The equation of continuity shows that a decrease in section must be accompanied by an increase in velocity. Thus at A, Fig. 50, the velocity is greater than at the orifice. Also the pressure in the liquid between the orifice and the vena contracta is greater than the pressure of the surrounding atmosphere, for according to Bernoulli's theorem an increase in velocity is accompanied by a decrease in pressure. Thus the pressure in the jet does not become atmospheric until the vena contracta is reached.

The seeming inconsistency of a pressure greater than the pressure of the surrounding atmosphere in an unconfined liquid disappears when we remember that particles moving in curvilinear paths must be acted upon by forces in a direction normal to the path and directed towards the center of curvature of the path so that the necessary centripetal acceleration may be imparted to them.* These forces are due to the greater pressure within the jet.

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* Martin, Text-book of Mechanics, Vol. II, page 90.

Coefficient of Contraction.—In order to compute the quantity of liquid discharged per unit time through an orifice the actual velocity of efflux at the vena contracta must be multiplied by the area of the cross-section of jet at the vena contracta. As the area of the orifice, A, is more easily found than the area at the vena contracta, the ratio of the area at the vena contracta to the area of the orifice has been determined experimentally. This ratio, denoted by c_c, is called the coefficient of contraction.

Coefficient of Discharge.—The discharge per unit of time through an orifice may be expressed by

$$(c_v\sqrt{2gh})(c_c A) = c_v c_c A\sqrt{2gh}.$$

The product of the coefficients c_v and c_c is called the coefficient of discharge and is denoted by c. This coefficient is for practical purposes much more important than either c_v or c_c. Its average value for a standard orifice is 0.60.

Exercise 89. Devise an experimental method for the determination of c without the individual determination of either c_v or c_c.

Applications.—In computing the discharge through an orifice it is customary to compute the velocity of efflux at the orifice (more accurately, at the vena contracta) neglecting all losses, and then multiply by the area of the orifice and by the coefficient of discharge. The last factor corrects not only for the loss in velocity, but also for the contraction in area.

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* Martin, Text-book of Mechanics, Vol. II, pp. 43 and 111.

As **an example** let us compute the discharge per second through an orifice whose area is one square inch and whose coefficient of discharge is 0.60. The water reaches the orifice through a pipe whose sectional area is 4 square inches. At a point 3 feet above the orifice a pressure gage reads 40 pounds per square inch.

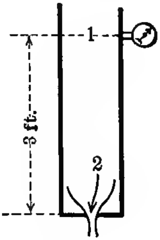


FIG. 51.

Fig. 51 illustrates the problem. The sections of the flow to consider are at 1 where the pressure is known and at 2 strictly at the vena contracta of the issuing jet. Practically this coincides with the orifice, for the distance between the orifice and the vena contracta equals about one-half the diameter of the orifice.

Let v_1 be the velocity of flow at section 1 and v_2 be the velocity of flow at section 2 when all resistances are neglected. Then by Bernoulli's theorem we have,

$$\frac{p_a+40\times 144}{62.4}+\frac{v_1^2}{2g}+3=\frac{p_a}{62.4}+\frac{v_2^2}{2g}+0.$$

The equation of continuity becomes

$$Q=\left(\frac{4}{144}\right)v_1=(0.60)\left(\frac{1}{144}\right)v_2.$$

From these equations we find

$$v_2=79.3 \text{ ft. per sec.},$$

and $$Q=0.33 \text{ cu.ft. per sec.}$$

In this solution the velocity of approach, v_1, has been taken into account. This velocity in the pipe leading to the orifice is 11.9 feet per second.

EXERCISE 90. In Fig. 52 the head is maintained constant and equal to h. Let A be the area of a horizontal section of the tank and a the area of the orifice. Show that the discharge per second equals

$$Q=ca\sqrt{2gh}\left\{\frac{1}{1-c^2\left(\frac{a}{A}\right)^2}\right\}^{\frac{1}{2}}.$$

EXERCISE 91. Compute the error, in per cent, due to neglecting the velocity of approach in Exercise 90 when the area of the tank section is 10 times the area of the standard orifice.

EXERCISE. 92. Compute the head lost due to flow through a standard orifice in terms of the actual velocity of discharge.

EXERCISE 93. Water flows through an orifice (area 2 sq.in., coefficient of discharge 0.59) into a caisson in which an absolute pressure of 3 atmospheres is maintained. A pressure gage attached to a 4-inch pipe leading to the orifice which is 100 feet below the gage reads 60 pounds per square inch. Compute the discharge per second and the velocity of efflux.

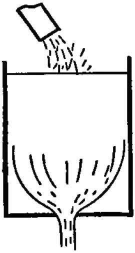


FIG. 52.

EXERCISE 94. An orifice, area a, in a vertical wall is h_1 feet under the free surface of the water on one side of the wall and discharges h_2 feet below the level of the tail water. Compute the discharge.

SECTION IX

FLOW UNDER CHANGING HEAD

When the head upon an orifice varies the velocity and the rate of discharge will also vary. In problems of this kind it is usually required to find the time in which a certain quantity of liquid is discharged.

The quantity of liquid that has left the tank in any interval of time dt can be computed in two ways:

1. By means of the area of a horizontal section of the tank and the drop in the free surface during the time dt.

2. By means of the area of the orifice, the coefficient of discharge, and the velocity of efflux which does not change during the time dt.

Equating the rates of discharge so computed we obtain a differential equation which can usually be solved for the time which elapses during the discharge of a certain quantity of liquid.

As **an example** of this method of procedure let us find the time required for the level of the free surface in tank 1, Fig. 53, to fall from h_1 to h_1' due to the flow into tank 2 through a submerged orifice.

Let the horizontal sectional areas of the tanks be constant and equal to A_1 and A_2, the area of the orifice, a, and let a fall of the surface in tank 1 of h_1-h_1' cause a rise of the free surface in tank 2 of $h_2'-h_2$.

Thus we have

$$A_1(h_1-h_1')=A_2(h_2'-h_2).$$

If at any instant during the flow the head on the orifice in tank 1 is y and in tank 2 the head is z, then the volume of liquid leaving tank 1 in the next dt seconds is

$$A_1\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and this must equal the volume entering tank 2, namely,

$$A_2\,dz.$$

At the instant considered the effective head on the orifice is $y-z$, the velocity of efflux is $c_v\sqrt{2g(y-z)}$ and the quantity discharged in the next dt seconds is

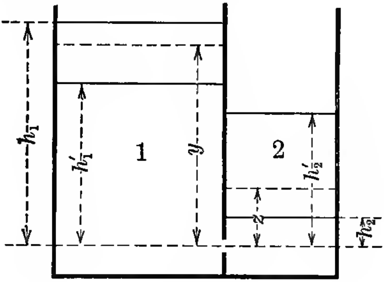


FIG. 53.

$$c\,a\sqrt{2g(y-z)}dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equating expressions (1) and (2) and solving for dt we have

$$dt=\frac{A_1\,dy}{c\,a\sqrt{2g\,(y-z)}}.$$

In order to integrate we must eliminate z. From Fig. 53 it is evident that

$$A_1(h_1-y)=A_2(z-h_2).$$

Thus
$$dt=\frac{A_1\,dy}{c\,a\sqrt{2g}\sqrt{y-\frac{A_1}{A_2}h_1+\frac{A_1}{A_2}y-h_2}}$$
$$=\frac{A_1\sqrt{A_2}}{c\,a\sqrt{2g}}\,\frac{dy}{\sqrt{(A_2+A_1)y-(A_1h_1+A_2h_2)}}.$$

This expression when integrated between the limits $y=h_1'$ and $y=h_1$ yields the required time

$$\frac{2\,A_1\sqrt{A_2}}{c\,a(A_2+A_1)\sqrt{2g}}\{\sqrt{A_2(h_1-h_2)}-\sqrt{(A_2+A_1)h_1'-(A_1h_1+A_2h_2)}\}.$$

Exercise 95. In how many seconds after the flow starts will the level in both tanks in the above example be the same?

Exercise 96. Show that a cylindrical tank will empty, through a small orifice in its bottom, in twice the time required to discharge the same volume of water when the head is maintained constant. The velocity of approach is to be neglected and c is to be assumed to remain constant.

Exercise 97. (*a*) Compute the time in which the head will fall from h_1 to h_2 in a prismatic tank due to discharge through a small orifice.

(*b*) From the result of (*a*) compute a hypothetical, mean velocity of discharge which remaining constant would cause the same quantity of liquid to pass through the orifice in the same time as it does under a falling head.

(*c*) Show that the result of (*b*) is the arithmetical mean of the velocities due to the heads h_1 and h_2.

Exercise 98. A hemispherical tank 10 feet in diameter and full of water is to be emptied through an orifice (coefficient of discharge 0.60) 2 inches in diameter. How many minutes will it take?

Exercise 99. A canal lock (area of base 600 square feet) contains water to a depth of 10 feet measured from the top of an orifice 1 foot square. The tail water stands at a constant depth of 2 feet above the base of the orifice. Compute the time it will take to "empty" the lock, ($c=0.60$).

Section X

LARGE ORIFICES LYING IN VERTICAL PLANES

All orifices so far considered were assumed so small relative to the head of water upon them that the differences in the velocities of flow through different portions of the orifice due to the differences in head could be neglected. We have assumed the velocity of flow to be that due to the head upon the center of the orifice.

As an example of a computation involving greater refinement let us find the flow through a **circular orifice** taking into account the variation of head on, and of velocity at, the various horizontal levels of the orifice.

As shown in Fig. 54 let h be the head at the center of the orifice and r the radius of the orifice. If y is measured positively downward from the center, the velocity through any horizontal differential strip is

$$\sqrt{2g(h+y)},$$

and the area of this strip is

$$2\sqrt{r^2-y^2}\,dy.$$

The flow neglecting contraction and friction will be

$$2\sqrt{2g(h+y)}\,\sqrt{r^2-y^2}\,dy.$$

If we represent the coefficient of discharge by c the discharge through the orifice is

$$Q = 2c\sqrt{2gh}\int_{-r}^{+r}\left(1+\frac{y}{h}\right)^{\frac{1}{2}}\sqrt{r^2-y^2}\,dy.$$

In order to integrate this expression $\left(1+\frac{y}{h}\right)^{\frac{1}{2}}$ must be expanded by the binomial theorem. When the terms

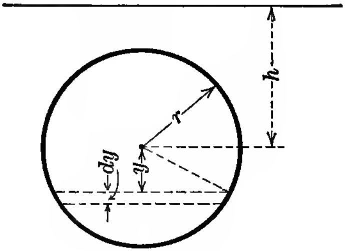


Fig. 54.

of the series so obtained are separately integrated, the following approximate value of the integral is obtained.

$$Q = 2c\sqrt{2gh}\int_{-r}^{+r}\left\{1+\frac{1}{2}\left(\frac{y}{h}\right)-\frac{1}{8}\left(\frac{y}{h}\right)^2+\dots\right\}\sqrt{r^2-y^2}\,dy$$

$$= 2c\sqrt{2gh}\left[\left(\frac{r^2}{2}\sin^{-1}\frac{y}{r}+\frac{1}{4}y\sqrt{r^2-y^2}\right)-\frac{1}{6h}(r^2-y^2)^{\frac{3}{2}}+\dots\right]_{-r}^{+r}$$

$$= 2c\sqrt{2gh}\left[\frac{r^2}{2}\pi+\dots\right.$$

$$= \pi r^2 c\sqrt{2gh}\left[1-\frac{1}{32}\left(\frac{r}{h}\right)^2-\frac{5}{1024}\left(\frac{r}{h}\right)^4-\frac{105}{65536}\left(\frac{r}{h}\right)^6\dots\right].$$

Exercise 100. Find the error made in assuming the velocity at the center of the orifice to be the velocity at all points if the head on the center equals four times the radius of the orifice.

Exercise 101. Show that the discharge through a rectangular orifice b feet wide having a head of h_1 feet on the top and h_2 feet on the bottom edge is

$$\tfrac{2}{3}cb\sqrt{2g}\{h_2^{\frac{3}{2}}-h_1^{\frac{3}{2}}\},$$

or if h is the head on the center of the orifice and a the height of the orifice that the discharge is

$$cba\sqrt{2gh}\left[1-\frac{1}{96}\left(\frac{a}{h}\right)^2-\frac{1}{2048}\left(\frac{a}{h}\right)^4-\dots\right].$$

Exercise 102. Compute the discharge through a triangular orifice (base, b, horizontal, vertex up) if the head on the vertex is h_1 and on the base is h_2.

Exercise 103. If the water approaches the orifice in Exercise 101 with a horizontal velocity u show that the effective head at the orifice is thereby increased by $h_u=\frac{u^2}{2g}$ and that the discharge becomes

$$Q=\tfrac{2}{3}cb\sqrt{2g}\{(h_2+h_u)^{\frac{3}{2}}-(h_1+h_u)^{\frac{3}{2}}\}.$$

CHAPTER V

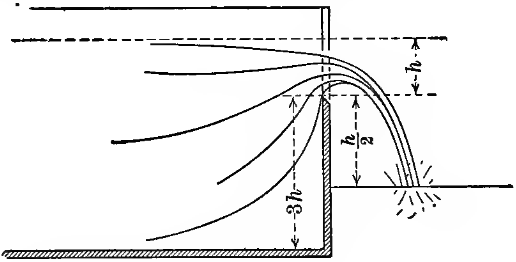
FLOW THROUGH WEIRS

INTRODUCTION

A WEIR is a notch cut in the upper part of a vertical plate which acts as a dam in obstructing the flow of water. The overflow through the notch or weir may be computed when the dimensions and the constants of the weir as well as the head of water over the weir crest are known.

In Fig. 55 are shown the plan and the section of a channel with a rectangular notch or weir at the right-hand end. The width of the weir is b. The head under which the flow through the weir occurs is not the depth of water over the crest or sill of the weir, for there is evidently a drop in the free surface at the weir as shown in Fig. 55. The head, h, is always measured from the crest to the level of the water in the channel so far up stream that the drop in the free surface due to the flow through the weir is negligible. The distance behind the weir at which h should be measured is about $6h$.

The flow through weirs is greatly affected by the contractions of the stream. The form of the stream leaving the weir is determined by four surfaces: the upper surface; the lower surface, whose form is determined by the crest contraction; and the lateral surfaces, whose forms are determined by the lateral or end contractions. It is very important that these contractions either be unrestricted or wholly suppressed when measurements of the flow are to be made by means of a weir. Some experimenters advise that the depth of the channel below the crest of the weir should be at least $3h$ in order that the crest contraction may fully develop. For unrestricted lateral contractions the distance between a lateral edge of the weir and the wall of the channel on its side should be $2h$ and even $3h$. Moreover the crest and sides of the weir should be sharp and the air should have free access to the lower surface of the stream.



SECTION THROUGH A-B

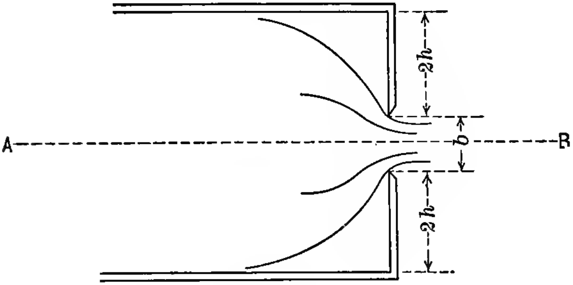


Fig. 55.

It is very probable that the flow through weirs depends upon the section of the stream at which contraction is complete (corresponding to the vena contracta of a jet) and the velocity attained at this point.

In practical experimentation it is, however, found impracticable to determine this section of greatest contraction and the formulas for the discharge through weirs are all based upon the dimensions b and h and involve empirical constants. In applying the formulas and constants of any experimenter great care must be taken to duplicate in every way the conditions under which the measurements for the determination of the empirical constants were made.

Section XI

RECTANGULAR WEIRS WITH SUPPRESSED END CONTRACTIONS

The Principle of Similarity.—This principle will be applied in the derivation of rational formulas for the discharge through weirs which serve as a basis for the practical formulas. The principle of similarity assumes that the stream lines in the flow of a frictionless liquid through similar and similarly placed orifices in similar vessels under similar heads will be similar. Under these conditions the sections of similarly placed tubes of flow taken at similar points will be proportional to the squares of any similar linear dimensions. Also the velocities at these points will be proportional to the square roots of the heads at these points. Therefore the flow across these sections, being in either case the product of area and velocity, will be proportional to the $\frac{5}{2}$ power of any linear dimensions.

Compare now the flow through a rectangular weir of width b under two different heads, Fig. 56 (*a*) and (*b*). Let one head be h and the other nh. To avoid the disturbing effect of the lateral contractions assume these contractions to be suppressed; that is, the sides of the channel of approach flush with the lateral edges of the weir. There is, however, sufficient depth of channel below the crest of the weir to permit of perfect crest contraction.

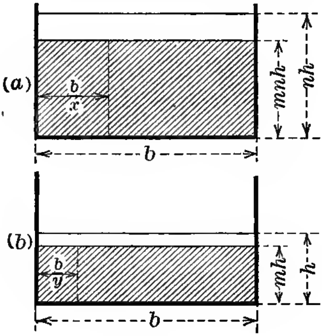


FIG. 56.

The sections of the streams in the plane of the weir and the projection upon this plane of the level of the water before the top contraction occurs are certainly not similar. If, however, the width at (*a*) be divided into x parts and the width at (*b*) into y parts, x and y may be so determined that the resulting parts will be similar.

The discharge through any part of weir (*a*) will be proportional to the product of the area of a section and the velocity due to the head over the weir crest; it will equal

$$K_1\left(\frac{b}{x}\right)(mnh)K_2(nh)^{\frac{1}{2}} = K\frac{b}{x}(nh)^{\frac{3}{2}}.$$

The total discharge through weir (*a*) will equal

$$K\,b(nh)^{\frac{3}{2}}.$$

As any part of weir (*b*) is similar to any part of weir (*a*) the same constant *K* will apply; the discharge through any part of weir (*b*) will equal

$$K\left(\frac{b}{y}\right)h^{\frac{3}{2}},$$

and the total discharge through weir (*b*) is

$$K\,b\,h^{\frac{3}{2}}.$$

The discharge through any rectangular weir with suppressed lateral contractions is thus proportional to the width of the weir and to the $\frac{3}{2}$th power of the head upon its crest, or the discharge equals

$$k\,b\,h^{\frac{3}{2}}.$$

Boussinesq's Theory * of the Discharge through a Rectangular Weir with Suppressed End Contractions.—When the end contractions are suppressed the stream lines in all planes perpendicular to the crest of the weir will be alike and resemble the curves shown in Fig. 57. Boussinesq considered the flow across the vertical plane *A C*, parallel to the weir crest and passing through the highest element of the surface due to the crest contraction.

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* Comptes Rendus, 105, 17. 1887.

Boussinesq assumed as the simplest hypothesis concerning the stream lines at the section DB that they are all normal to the plane BD and that they all have the same center of curvature C.

Let the radius of curvature of the lowest stream line at B be denoted by R_0, then the radius of curvature of the stream line at any point A, will be

$$R = R_0 + z - e.$$

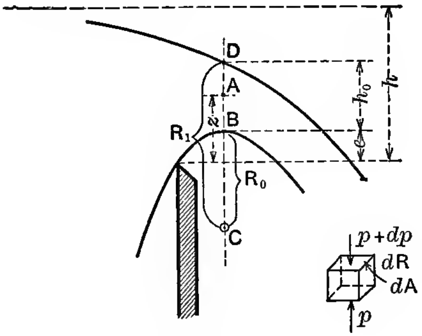

FIG. 57.

The pressure at B and at D must be atmospheric, but the pressure varies between these points. Let p be the pressure at A.

At the lower right hand of Fig. 57 is shown an element of the water at A. This element is assumed to move in a circle of radius R with a horizontal velocity v, therefore the sum of all vertical forces acting upon it must produce the necessary centripetal acceleration $\frac{v^2}{R}$. These vertical forces are (downward positive) $dp\,dA$ due to the hydrostatic pressure p, and $w\,dA\,dR$ due to gravitation;

thus $$dp\,dA+w\,dA\,dR=\left(\frac{w\,dA\,dR}{g}\right)\left(\frac{v^2}{R}\right)$$

or $$\frac{dp}{dR}=\frac{w}{g}\frac{v^2}{R}-w. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Another equation involving p and R may be obtained by means of Bernoulli's theorem applied to a tube, bounded by stream lines, extending from a point on the free surface of still water to the point A. With the horizontal plane through the crest as datum plane

we have, $$\frac{p}{w}+\frac{v^2}{2g}+z=\frac{p_a}{w}+h.$$

Differentiating $$\frac{dp}{w}+\frac{vdv}{g}+dz=0,$$

or $$\frac{dp}{dz}=-\frac{w}{g}\frac{vdv}{dz}-w. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equating the right-hand members of Equations (1) and (2) and eliminating z by means of the relation

$$R=R_0+z-e \quad \text{or} \quad dR=dz,$$

we have, $$\frac{v^2}{R}=-\frac{vdv}{dR}.$$

Integrating, we have,

$$\log R=-\log v+\log C,$$

or $$Rv=C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

To determine C we must find the velocity at B by Bernoulli's theorem, for at this point the radius of curvature R_0 is assumed as known.

Thus $$\frac{p_a}{w}+0+h=\frac{p_a}{w}+\frac{v_0^2}{2g}+e,$$

whence $$v_0=\sqrt{2g(h-e)}.$$

From Equation (3),

$$C=R_0\, v_0=R_0\sqrt{2g(h-e)},$$

and thus $$v=\frac{C}{R}=\sqrt{2g(h-e)}\,\frac{R_0}{R_0+z-e}.$$

The discharge through the weir if its width is b, is therefore

$$Q=\int_{z=e}^{z=h_0+e}\sqrt{2g(h-e)}\frac{R_0}{R_0+z-e}\,b\,dz$$

$$=b\,R_0\sqrt{2g(h-e)}\Big[\log\,(R_0+z-e)\Big]_{z=e}^{z=h_0+e}$$

$$=b\,R_0\sqrt{2g(h-e)}\,\log\frac{R_0+h_0}{R_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Bazin found experimentally that $\frac{e}{h}=0.13$, so that e may be assumed as known when h is known, h_0 can be determined only by means of refined experimentation, and R_0 cannot be determined experimentally. In order to eliminate h_0 and R_0 Boussinesq made use of the following principle due to Belanger.

When steady flow exists in an open channel the heads at all points are as low as is consistent with the necessary velocities and with the law of continuity.

Thus the head h during steady flow will become as low as possible consistent with a given discharge Q, or, for a given h the discharge Q must be a maximum.

Assuming h to remain constant Equation (4) contains two variables h_o and R_o, so that before the conditions for a maximum value of Q can be determined R_o must be expressed in terms of h_o or vice versa. Better still both of these variables may be expressed in terms of a third variable with the advantage of simplifying the expression algebraically. We see from Equation (3) that

$$R\,v = \text{a constant},$$

thus $$R_1\,v_1 = R_o\,v_o,$$

and $$y = \frac{v_1}{v_o} = \frac{R_o}{R_1},$$

where y is the third variable in terms of which both h_o and R_o are to be expressed.

As $v_1 = \sqrt{2g(h-e-h_o)}$ and $v_o = \sqrt{2g\,(h-e)}$ by reason of Bernoulli's theorem applied between the free still surface and D, and the free still surface and B, respectively,

we have $$y^2 = \frac{v_1{}^2}{v_o{}^2} = \frac{h-e-h_o}{h-e},$$

whence $$h_o = (h-e)(1-y^2).$$

Also as, $$y = \frac{R_o}{R_1} = \frac{R_o}{R_o+h_o},$$

we have $$R_o = \frac{y\,h_o}{1-y} = (h-e)\,y\,(1+y).$$

Q as given by Equation (4) can now be expressed in terms of the single variable, y, as follows:

$$Q = b\sqrt{2g}(h-e)^{\frac{3}{2}}(y+y^2)\log\frac{1}{y}.$$

Let $$Y = (y+y^2)\log\frac{1}{y}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

then Q is proportional to Y, i.e., Q reaches a maximum when Y reaches a maximum and Q is positive when Y is positive, etc.

Now Q and therefore Y must always be positive. This occurs only for values of y lying between 0 and 1. Moreover Y, and therefore Q, becomes zero when $y=0$ and when $y=1$.

Exercise 104. Prove these last statements.

To obtain the value of y which yields a maximum value of Y put

$$\frac{dY}{dy}=(1+2y)\log\frac{1}{y}-(1+y)=0, \quad . \quad . \quad . \quad . \qquad (6)$$

and solve this equation for y.

A root of Equation (6) is

$$y=0.4685. \ . \ . \ . \ .$$

Exercise 105. Check this statement.

The second derivative of Y is

$$\frac{d^2Y}{dy^2}=-\left[3+\frac{1}{y}+2\log y\right],$$

and this second derivative is negative for all values of y lying between 0 and 1.

Exercise 106. Show this by computation for $y=\frac{1}{100}, \frac{1}{10}, 1$.

Therefore Equation (6) has only one root between $y=0$ and $y=1$, and this root yields a maximum value of Q, which is

$$Q=0.5216\, b\sqrt{2g}\,(h-e)^{\frac{3}{2}}.$$

EXERCISE 107. Deduce this value of Q and show that $h_0=0.7805(h-e)$ and $R_0=0.6881(h-e)$ when the discharge reaches a maximum value for a given h.

If we write Q in the form

$$Q=c\,b\,h\sqrt{2gh},$$

the coefficient of discharge becomes

$$c=0.5216\left(1-\frac{e}{h}\right)^{\frac{3}{2}}.$$

Bazin * in his experiments made in 1886 at the request of Boussinesq found that $\frac{e}{h}=0.13$ when the crest contraction is fully developed.

Thus $$c=0.423.$$

Bazin's Empirical Formula † for the discharge per second through a rectangular weir with end contractions suppressed is

$$Q=c\,bh\sqrt{2gh},$$

wherein when $h=0.164,\ 0.328,\ 0.656,\ 0.984,\ 1.312,\ 1.64,\ 1.968$ feet,
$c=0.448,\ 0.432,\ 0.421,\ 0.417,\ 0.414,\ 0.412,\ 0.409,$

or $$c=0.405+\frac{0.00984}{h}, \text{ approximately,}$$

these values were determined on a weir 6.56 feet wide.

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* Comptes Rendus, 105, 212. 1887.

† Annales des Ponts et Chaussées (Vol. II), 445. 1891.

Rafter * at Cornell made experiments on a weir identical with Bazin's and found for

h=0.1, 0.5, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0 feet,

c=0.4286, 0.4230, 0.4174, 0.4106, 0.4094, 0.4112, 0.4133, 0.4136.

Fteley and Stearns' Formula † is

$$Q=3.31\ b\ h^{\frac{3}{2}}+0.007\ b.$$

Exercise 108. If the added effect due to the width of the weir is neglected what is the coefficient of discharge c in Fteley and Stearns' formula?

Francis' Formula.‡—James B. Francis at Lowell, Mass., in 1851 made extensive experiments with large weirs in which b averaged 10 feet and h varied between 0.6 and 1.6 feet. He deduced the formula

$$Q=3.33\ b\ h^{\frac{3}{2}}.$$

Exercise 109. Compute c from Francis' formula.

Exercise 110. It has been proposed to regard a rectangular weir as a large rectangular orifice of height h with the upper edge level with the still water behind the orifice. Returning to Exercise 101 deduce from it a weir formula, and using Francis' results determine c.

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* Trans. A. S. C. E., Vol. XXVII.

† Trans. A. S. C. E., 12, 1. 1883.

‡ Lowell Hydraulic Experiments.

SECTION XII

RECTANGULAR WEIRS WITH END CONTRACTIONS

The effect of end contractions evidently lies in the reduction of the effective width of the weir. The flow over the weir may be divided into three parts by planes normal to the crest (Fig. 58). The middle part of width b_1 may be assumed to be unaffected by the end contractions, while the effect of these contractions is felt throughout the widths b_2.

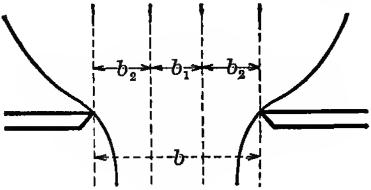

FIG. 58.

Experiments show that the width b_2 is proportional to the head h, so that

$$b_2 = m\,h,$$

where m is a constant.

The flow through the central part of the stream is

$$k_1\,b_1\,h^{\frac{3}{2}},$$

and through each lateral part

$$k_2\,b_2\,h^{\frac{3}{2}},$$

where k_1 and k_2 are constants, $k_2 < k_1$.

The total discharge is thus

$$Q = k_1\,b_1\,h^{\frac{3}{2}} + 2\,k_2\,b_2\,h^{\frac{3}{2}}.$$

From Fig. 58, $b = b_1 + 2b_2$ or $b_1 = b - 2b_2$.

Therefore $$Q = \{k_1(b - 2b_2) + 2k_2\, b_2\} h^{\frac{3}{2}}.$$
$$= \{k_1\, b + 2(k_2 - k_1) b_2\} h^{\frac{3}{2}}.$$

and as $b_2 = m\, h$,

we have, $$Q = k_1 \left\{ b + \frac{2m\,(k_2 - k_1)}{k_1} h \right\} h^{\frac{3}{2}},$$

or $$Q = k_1 \{ b - 2k_3\, h \} h^{\frac{3}{2}},$$

where k_1 and k_3 are constants to be determined by experiment.

Francis in his experiments obtained as average values

$$k_1 = 3.33 \quad \text{and} \quad k_3 = 0.1,$$

so that **Francis' Formula** for the discharge per second through weirs with two end contractions may be written

$$Q = 3.33 \left\{ b - \frac{2h}{10} \right\} h^{\frac{3}{2}}.$$

Exercise 111. A rectangular weir with one end-contraction suppressed and 8 feet wide discharges under a head of 1.5 feet. Compute the discharge.

Exercise 112. A weir in a dam must allow the passage of a possible rainfall of 2 inches in 24 hours on a water shed of 10 square miles. How wide should the weir be made if the water is not to rise more than 2 feet above its crest?

SECTION XIII

THE VELOCITY OF APPROACH

All formulas given above are derived under the assumption that h is the vertical distance between the crest of the weir and the level of the still-water surface above the weir. In all practical applications even though h is measured to the free surface some 6 to 15 feet behind the weir the water at this point always has some appreciable velocity. This is the velocity of approach. If the channel is very wide and very deep the velocity of approach may be so small that its effect is negligible. Otherwise the velocity of approach causes the discharge through the weir computed by means of the measured head to be less than the actual discharge.

Of course, if Q is the actual discharge through the weir and A is the cross-section of the channel at which the velocity of approach is u we have,

$$u=\frac{Q}{A}.$$

Let b represent the width of the weir;

h, as before, the vertical distance between the crest of the weir and still water;

h_1, the vertical distance between the crest and the surface of the water (say 10 or 15 feet above the weir) as measured;

A, the cross-section of the channel and

u, the mean velocity of approach both at the point to which h_1 is measured;

c, the coefficient when there is no velocity of approach;

c_1, the coefficient when h_1 is the head used so that there exists a velocity of approach.

Then the discharge through the weir is

$$Q = c\,b\,h\sqrt{2gh},$$

or when the measured value h_1 is used

$$Q = c_1\,b\,h_1\sqrt{2gh_1}.$$

The mean velocity of approach is thus

$$u = \frac{Q}{A} = \frac{c_1\,b\,h_1\sqrt{2gh_1}}{A}.$$

The head, $h - h_1$, producing this mean velocity of approach is

$$h - h_1 = \alpha\frac{u^2}{2g} = \alpha\frac{c_1^2\,b^2\,h_1^3}{A^2}.$$

Where α is some constant to be determined experimentally.
Bazin found α to be 1.66,
Rafter assumed α as 1.00,
Fteley and Stearns found α to lie between 1.45 and 1.50.

Thus $$h = h_1 + \frac{\alpha\,c_1^2\,b^2\,h_1^3}{A^2} = h_1\left\{1 + \frac{\alpha\,c_1^2\,b^2\,h_1^2}{A^2}\right\},$$

and $$Q = c\,b\,h_1\sqrt{2gh_1}\left\{1 + \frac{\alpha\,c_1^2\,b^2\,h_1^2}{A^2}\right\}^{\frac{3}{2}}.$$

As $\sqrt{\alpha\,c_1}\,b\,h_1$ is smaller than A we find after expanding by means of the binomial theorem that

$$Q = c\,bh_1\sqrt{2gh_1}\left\{1 + \frac{3}{2}\,\frac{\alpha\,c_1^2\,b^2\,h_1^2}{A^2}\right\}, \text{ approximately.}$$

Placing $$\frac{3\,\alpha\,c_1^2}{2} = \beta,$$

we have $$Q=c\,b\,h_1\sqrt{2gh_1}\left\{1+\frac{\beta\,b^2\,h_1^2}{A^2}\right\}.$$

EXERCISE 113. Show that

$$Q=c\,b\,h_1\sqrt{2gh_1}\left\{1+\frac{\beta\,h_1^2}{(h_1+p)^2}\right\},$$

(where p = the height of the crest above the floor of the channel) as used by Bazin for suppressed weirs.

Bazin found that the mean value of β is 0.55, so that in terms of h_1 **Bazin's formula for suppressed weirs becomes**

$$Q=\left(0.405+\frac{0.00984}{h_1}\right)\left\{1+\frac{0.55h_1^2}{(h_1+p)^2}\right\}b\,h_1\sqrt{2gh_1}.$$

Francis' Formula when correction is made for the velocity of approach is

$$Q=3.33\left(b-\frac{2h_1}{10}\right)\left[(h_1+h_u)^{\frac{3}{2}}-h_u^{\frac{3}{2}}\right],$$

where h_u is the head producing the velocity of approach u, thus

$$h_u=\frac{u^2}{2g}.$$

EXERCISE 114. Show how Francis' formula can be obtained from Exercise 103.

In order to apply this formula an approximate value of Q is found by neglecting h_u and u. Then an approximate value of u is computed by means of $\frac{Q}{A}$. This value of u permits an approximate computation of h_u.

Now Q is recomputed using this value of h_u. This second approximation of Q is usually sufficiently accurate.

Exercise 115. If the weir described in Exercise 111 is supplied by a channel whose section at the point where h_1 is measured is 10×5 feet, what is the approximate velocity of approach?

Recompute the discharge allowing for this velocity of approach.

Section XIV

TRIANGULAR WEIRS

A triangular notch having its sides equally inclined to the vertical is sometimes used to measure small quantities of water.

As the conditions of flow through a triangular weir are similar under all heads the experimental coefficients are more likely to remain constant under varying head with this form of weir than with the rectangular weir.

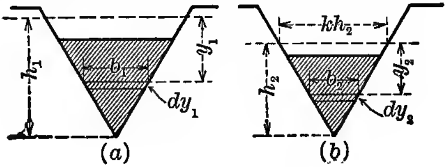

Fig. 59.

The development of the formula for Q, the discharge per second, may be based upon the principle of similarity.

As in Fig. 59 consider a weir through which a liquid flows first under a head h_1 and then under a head h_2. The discharge through any horizontal strip of width b_1 and differential depth will be proportional to

$$(b_1\,dy_1)\sqrt{2g\,y_1},$$

or

$$Q_1 = c\sqrt{2g}\int b_1\sqrt{y_1}\,dy_1.$$

If Figs. 59 (*a*) and (*b*) are considered in every way similar then the discharge through (*b*) will be

$$Q_2=c\sqrt{2g}\int b_2\sqrt{y_2}\,dy_2.$$

But $\quad b_1=nb_2, \quad y_1=ny_2.$

Therefore

$$Q_1=c\sqrt{2g}\int n\,b_2\sqrt{ny_2}\,ndy_2$$

$$=c\sqrt{2g}\,n^{\frac{5}{2}}\int b_2\sqrt{y_2}\,dy_2.$$

So that $\quad Q_1=n^{\frac{5}{2}}\,Q_2.$

As $\quad y_2=zh_2,$

where z is a variable $(1 \geqq z > 0)$,

and $\quad b_2=kh_2\dfrac{h_2-y_2}{h_2},$

where k is some constant,

$$Q_2=c\sqrt{2g}\int kh_2\left(\frac{h_2-zh_2}{h_2}\right)\sqrt{z}\sqrt{h_2}\,h_2d\,z,$$

$$=c\,k\sqrt{2g}\,h_2^{\frac{5}{2}}\int(1-z)\,\sqrt{z}\,d\,z.$$

So that $\quad Q_2=C\,h_2^{\frac{5}{2}},$

and $\quad Q_1=n^{\frac{5}{2}}\,C\,h_2^{\frac{5}{2}}=C\,h_1^{\frac{5}{2}},$

or for any head, h,

$$Q=C\,h^{\frac{5}{2}}.$$

Thomson found that for a triangular weir having sharp edges and an angle of 90 degrees at the vertex, C was practically constant for all heads and that

$$C=2.635,$$

when h is measured in feet and Q in cubic feet per second.

Exercise 116. Show that

$$Q=\frac{8}{15}c\tan\alpha\sqrt{2g}\,h^{\frac{5}{2}},$$

for a triangular weir (angle at the vertex, 2α) when the weir is considered as a large orifice.

Point out three objections to this treatment of the problem.

Exercise 117. A triangular weir (angle=90°) discharges water under a head of 2.4 feet. Compute the discharge in cubic feet and in gallons per second.

Trapezoidal Weir of Cippoletti.—Cippoletti proposed the use of trapezoidal weirs. The sides were to slope at an angle just sufficient to counteract the effect of the end contractions in a rectangular weir. Thus the trapezoidal weir, Fig. 60, when its sides have the proper slope, would discharge at the same rate as a suppressed rectangular weir of width b.

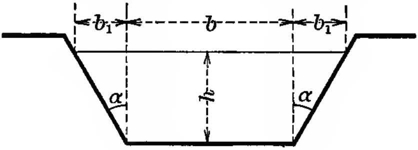

Fig. 60.

Exercise 118. Show that the slope of the sides must be

$$\tan\alpha=\tfrac{1}{4},$$

if the coefficient in Francis' formula and in the formula deduced in Exercise 115 are considered equal.

Cippoletti found as the results of his experiments on weirs, proportioned as above explained, that

$$Q = 3.367\, b\, h^{\frac{3}{2}}.$$

Flinn and Dyer found as a result of their experiments that the constant should be

$$3.283.$$

They corrected for the velocity of approach by adding $1.4\frac{u^2}{2g}$ to h while Cippoletti used Francis' method of correction.

CHAPTER VI

FLOW THROUGH PIPES

SECTION XV

THE HYDRAULIC GRADIENT

THE fundamental equation by means of which the flow of liquids is studied is Bernoulli's equation

$$\frac{p}{w}+\frac{v^2}{2g}+h=\text{constant.}$$

The equation as here written expresses the fact that the total energy of a unit weight of the liquid remains constant. It should of course be understood that in this form the equation takes no account of losses due to friction, enlargements, etc., or of the energy supplied to (or withdrawn from) the liquid by pumps (or by turbines).

In Fig. 61 is shown a pipe line starting at a reservoir so large that the discharge through the pipe does not materially lower the level of the free surface. A valve at the lower end of the pipe throttles the flow and steady flow is assumed to exist.

The energy due to the pressure at any point of the pipe, $\frac{p}{w}$, may conveniently be written $\frac{p-p_a}{w}+\frac{p_a}{w}$. Here $\frac{p_a}{w}$ represents the head due to atmospheric pressure and $\frac{p-p_a}{w}$ the head corresponding to the gage pressure or the height to which the liquid would rise in an open piezometer tube attached to the pipe at the point in question. As shown in Fig. 61 the sum of the pressure head, the velocity head, and the position head is constant and equals $h_1+\frac{p_a}{w}$. The line joining the tops of the liquid columns in the piezometer tubes is called the **hydraulic gradient.** An enlargement of the pipe causes a decrease in the velocity of flow, therefore a decrease in the velocity head, and the hydraulic gradient rises.

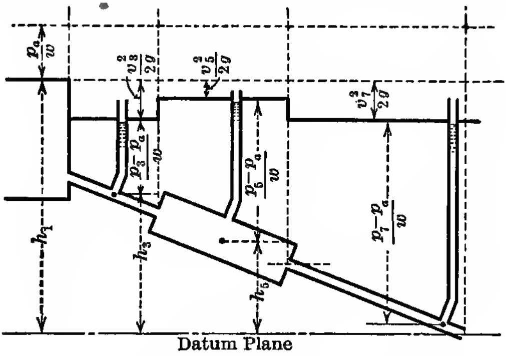

FIG. 61.

In this chapter the effect of the losses of energy which occur during the flow of liquids in pipes will be studied. These losses are due to:

(*a*) the loss of head at the entrance to the pipe;
(*b*) the losses due to friction;
(*c*) the losses due to sudden enlargements;
(*d*) the losses due to sudden contractions;
(*e*) the losses due to bends;
(*f*) the losses due to obstacles, such as valves.

The hydraulic gradient shown in Fig. 61 is the ideal gradient obtained when all losses are neglected.

When the losses of energy are taken into account Bernoulli's equation between any two points would read

$$\frac{p_1}{w}+\frac{v_1^2}{2g}+h_1=\frac{p_2}{w}+\frac{v_2^2}{2g}+h_2+h_l$$

where h_l represents the loss in energy per unit weight of liquid during its passage from section 1 to section 2. This loss is due to the sum of the effects of all the resistances encountered during the flow.

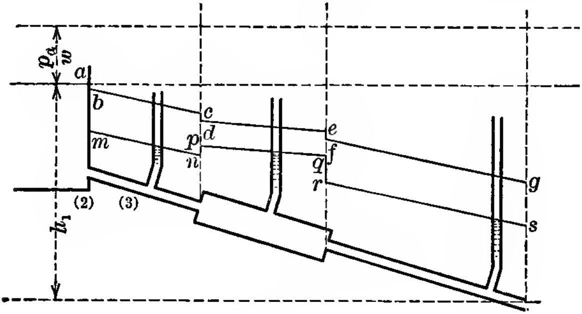

FIG. 62.

The actual hydraulic gradient of the pipe line shown in Fig. 61 when the losses in energy are taken into account is roughly indicated in Fig. 62. Instead of starting with the level of the free surface in the reservoir and subtracting the velocity head in order to obtain a point on the hydraulic gradient we must now first establish a new zero line by subtracting the heads due to the losses from the level of the free surface in the reservoir. These losses are cumulative and thus the zero line departs more and more from the level of the water in the reservoir as the length of the pipe increases.

The first loss occurs at the entrance to the pipe as indicated by the drop *a b*, Fig. 62. The friction in the first length of pipe causes a further, gradually increasing, loss of head as shown from *b* to *c*. The sudden enlargement of the pipe diameter (which is to be avoided in practice) causes the sudden loss of head indicated by the drop *c d*. As the frictional losses in a pipe of larger diameter are less than the losses in a pipe of smaller diameter the slope of the line *d e* is less than the slope of *b c*.

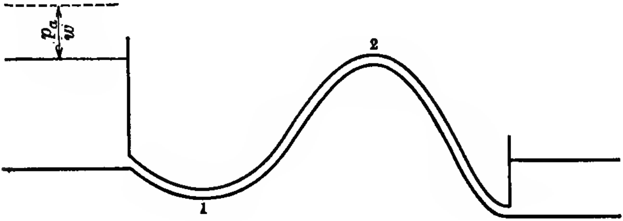

FIG. 63.

From this zero line we may now subtract the velocity heads and obtain the hydraulic gradient *m n p q r s*, which passes through the tops of the columns of water in the (imaginary) piezometer tubes placed along the pipe line.

EXERCISE 119. Draw the hydraulic gradient in Figs. 61 and 62 when the valve at the lower end of the pipe is closed.

EXERCISE 120. Sketch the hydraulic gradient for the pipe line shown in Fig. 63. The velocity with which the water enters the lower reservoir is wasted by impact and in the production of eddies so that the hydraulic gradient at its lower end may be assumed to reach the level of the free surface in the lower reservoir. Sketch piezometer tubes at 1 and 2.

Loss at Entrance to Pipe.—The loss of head at the entrance to a pipe line depends upon the nature of the inlet to the pipe. If the end of the pipe is flush with the side of the reservoir and leaves it normally the loss is $0.5\frac{v^2}{2g}$ where v is the velocity in the pipe. For a re-entrant pipe with sharp edges the coefficient may be as high as 1.3, while for a bell-mouthed pipe it may be as low as 0.02.

SECTION XVI

THE LOSS OF HEAD DUE TO FRICTION

Fundamental Equations.—The resistances encountered by a liquid flowing in a pipe depend not only on the viscosity of the liquid but also upon the roughness of the pipe. The flow in practical cases is turbulent, due to the formation of eddies. On the whole the conditions involved are so complicated that a complete rational discussion is impossible. The following simple theory leads to an approximate formula which has been found to satisfy all practical requirements provided the experimental coefficients are judiciously chosen.

If Bernoulli's equation is applied to two sections in a pipe line we have

$$\frac{p_1}{w}+\frac{v_1^2}{2g}+h_1=\frac{p_2}{w}+\frac{v_2^2}{2g}+h_2+h_f,$$

where h_f represents the head (energy per pound of liquid) lost in friction between sections 1 and 2. If we assume the diameter of the pipe to be constant then $v_1 = v_2$, and

$$h_f = \left(\frac{p_1}{w} + h_1\right) - \left(\frac{p_2}{w} + h_2\right).$$

To introduce the effect of the frictional forces acting upon the liquid between the sections 1 and 2, make a free body of the liquid between sections 1 and 2, Fig. 64.

Let A be the sectional area of the pipe and l the distance between the sections 1 and 2, then the forces acting upon

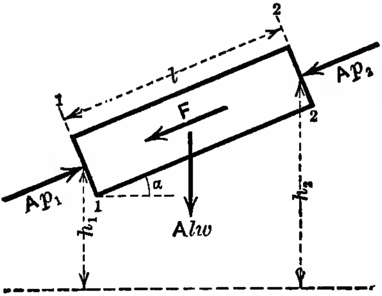

FIG. 64.

the portion of the liquid considered are, its weight $A\,l\,w$, the forces Ap_1 and Ap_2 on the end sections, the forces due to the pressure of the pipe normal to the cylindrical surface (not shown in the figure), and the frictional forces, the combined action of which is represented by F.

As the liquid as a whole moves with constant velocity no acceleration of the water occurs and the sum of all forces in the direction of the axis of the pipe may be put equal to zero.

Thus $$Ap_1 - Ap_2 - A\,w\,l \sin\alpha - F = 0.$$

As $$l \sin \alpha = h_2 - h_1 \quad \text{(Fig. 64)},$$

we have, $$\frac{p_1}{w} - \frac{p_2}{w} + h_1 - h_2 = \frac{F}{Aw},$$

so that $$h_f = \frac{F}{Aw}$$

If f' represents the frictional resistance per unit area of the surface of contact between the liquid and the pipe and d, the internal diameter of the pipe,

we have, $$F = f'(\pi\, d\, l),$$

and $$h_f = \frac{f'(\pi\, d\, l)}{\frac{\pi\, d^2}{4} w} = \frac{4l}{wd} f'.$$

Experiments have shown that f' varies approximately as the square of the velocity, v, so that

$$f' = k\, v^2, \text{ where } k \text{ is a constant,}$$

and $$h_f = \frac{4\, l\, k}{wd} v^2.$$

As this expression represents a head and must be expressed in units of length; the constant k cannot be a numerical coefficient, having no dimensions.

EXERCISE 121. What are the dimensions of k?

It is more convenient to place the constant $\frac{k}{w} = \frac{f}{2g}$ and write

$$\mathbf{h_f = f \frac{4l}{d} \frac{v^2}{2g}},$$

where f is a numerical coefficient, called the friction factor, to be determined by experiment.

EXERCISE 122. Explain an experimental method for the determination of f.

The Chezy Formula, one of the oldest of the formulas used for both open or closed channels, involves a factor termed the **mean hydraulic depth.**

The **wetted perimeter** is defined as that part of the periphery of the cross-section of a channel or pipe which is in contact with the liquid.

The **mean hydraulic depth** is defined as the area of the cross-section of the stream divided by the wetted perimeter.

The Chezy formula is

$$v = C\sqrt{m\,s},$$

where m = the mean hydraulic depth;

s = the "slope of the hydraulic gradient," defined as $\frac{h_f}{l}$, and

C = a coefficient which increases with m and with v

EXERCISE 123. Show that the Chezy formula is simply another form of

$$h_f = f\frac{4l}{d}\,\frac{v^2}{2g}.$$

EXERCISE 124. What relation exists between f and C in the above formulas?

The reason for not including the 4 in the friction factor f is now apparent. The formula

$$h_f = f\frac{4l}{d}\,\frac{v^2}{2g},$$

may be written

$$h_f = f\frac{l}{m}\,\frac{v^2}{2g},$$

for the mean hydraulic depth of a pipe flowing full is

$$m=\frac{\frac{\pi d^2}{4}}{\pi d}=\frac{d}{4}.$$

In the **application of the formula** containing the friction factor, f, which varies (with the nature of the interior surface of the pipe, the internal diameter of the pipe, and the velocity of flow) between 0.0152 and 0.0028, it is customary to assume $f=0.0050$ in preliminary calculations. This will lead to approximate values of v or d which will allow a value of f, more suitable to the problem in hand, to be selected from the tabulated results of experiments. The recomputation of v or d with this new value of f will then lead to closer approximations of v or d.

Thus, to find the velocity of flow and the distribution of head during the discharge through a new cast-iron pipe 2000 feet long and 10 inches in diameter from a reservoir affording a head of 100 feet above the discharging end of the pipe proceed as follows:

The total available head, 100 feet, will be used:

1, to produce the velocity of discharge v;
2, to overcome the resistance at the entrance to the pipe;
3, to overcome the frictional resistance in the pipe.

Applying Bernoulli's equation

$$\frac{p_1}{w}+\frac{v_1^2}{2g}+h_1=\frac{p_2}{w}+\frac{v_2^2}{2g}+h_2+h_f$$

between the free surface of the reservoir and the discharging end of the pipe line we have,

$$\frac{p_a}{w}+\frac{0^2}{2g}+100=\frac{p_a}{w}+\frac{v^2}{2g}+0+.5\frac{v^2}{2g}+0.005\frac{4(2000)}{\frac{10}{12}}\frac{v^2}{2g}$$

whence, $$(1+.5+48)\frac{v^2}{2g}=100,$$

$$v^2=130,$$

$$v=11.4 \text{ feet per sec.}$$

The tabulated values of f show that for a 10-inch cast-iron pipe with water flowing at 11.4 feet per second $f=.0051$. Therefore recomputing

$$(1+.5+49)\frac{v^2}{2g}=100,$$

$$v^2=127.3,$$

$$v=11.3 \text{ feet per sec.}$$

Note that of the total head of 100 feet, $\frac{v^2}{2g}=1.98$ feet are available in the form of velocity at the discharging end of the pipe, $.5\frac{v^2}{2g}=0.99$ foot is lost at entrance to pipe, and $49\frac{v^2}{2g}=97.0$ feet are lost in overcoming frictional resistances.

Exercise 125. Compute the diameter of a pipe 1000 feet long whose discharging end is 5 feet lower than the surface of the reservoir supplying it with water and which is to discharge 2000 gallons per minute.

SECTION XVII

APPLICATIONS

The Hydraulic Transmission of Energy.—Energy may be transmitted with very high efficiency by means of water under pressure. Each pound of water under a pressure of p pounds per square foot contains $\frac{p}{w}$ foot-pounds of energy. This energy may be utilized in operating cranes, elevators, etc., in which pistons or plungers are driven by the water. The velocity of flow during such transmission is relatively low and the kinetic energy is not utilized.

EXERCISE 126. Water is delivered to a horizontal pipe l feet long, internal diameter, d feet, at a pressure of p pounds per square foot. If the velocity of flow is v feet per second, compute the efficiency of transmission.

EXERCISE 127. A pipe, 8 inches in internal diameter and 2000 feet long is supplied with water at a pressure of 1000 pounds per square inch, gage. The velocity of flow is 4 feet per second; under these conditions $f=0.00582$. Compute the efficiency of transmission and the power available at the far end of the pipe.

EXERCISE 128. The greater the velocity the more water and therefore the more energy can be transmitted through a given pipe line. Show that the pressure energy transmitted per second through a long pipe is a maximum when one-third of the energy supplied is wasted in friction.

EXERCISE 129. Show that the horse-power transmitted through a pipe equals

$$1.25\eta\, d^{\frac{5}{2}}\, p^{\frac{3}{2}}\sqrt{\frac{1-\eta}{fl}},$$

where η = the efficiency;
l = the length of the pipe, in feet;
d = the internal diameter of the pipe, in feet;
f = the friction factor;
p = the initial pressure in the pipe, in pounds per square inch.

Pipe Line Terminating in a Nozzle.—Nozzles transform pressure into velocity so that a jet having high velocity is delivered. The kinetic energy of jets is utilized in the development of power by means of impulse wheels and the momentum of jets is used in hydraulic mining.

The loss due to the resistance encountered in a nozzle is computed from its coefficient of velocity. This coefficient varies between 0.95 and 0.98. The coefficient of contraction of a properly designed nozzle is unity.

If v is the actual velocity with which the water leaves a nozzle whose coefficient of velocity is c_v, the velocity which the water would attain if all resistances could be eliminated would be $\frac{v}{c_v}$. Thus the kinetic energy per pound of water lost during the passage through the nozzle is

$$\frac{1}{2g}\left(\frac{v}{c_v}\right)^2-\frac{v^2}{2g}=\left(\frac{1}{c_v^2}-1\right)\frac{v^2}{2g},$$

where v is the actual velocity of discharge.

EXERCISE 130. A nozzle (diameter d feet, coefficient of velocity c_v) discharges under a head of h feet. The water reaches the nozzle through a pipe l feet long and D feet in diameter. Write an equation from which v, the velocity of discharge, may be found.

Exercise 131. A pipe l feet long and D feet in diameter is to be fitted with a nozzle. The kinetic energy developed from a head of h feet is to be used in driving an impulse wheel. What must be the diameter at the end of the nozzle so that a maximum of kinetic energy is developed? It is permissible to neglect the losses at the entrance to the pipe and in the nozzle, a long pipe line being assumed.

Exercise 132. Under the conditions established in Exercise 131 what fraction of the total head is lost in pipe friction? (Compare with Exercise 128.)

Exercise 133. In hydraulic mining the pressure exerted by a jet of water is utilized to disintegrate the gold-bearing strata. The pressure exerted upon a surface by a jet is proportional to the momentum of the moving water. Compute the diameter of a nozzle which will produce a jet of maximum momentum under the conditions of Exercise 131.

Exercise 134. What fraction of the available head is lost in pipe friction under the conditions established in Exercise 133?

The Siphon.—Whenever a pipe line rises above its hydraulic gradient the pressure within the pipe becomes less than atmospheric. Under these conditions the air dissolved in the water and the air which may leak into the pipe tends to collect at the summit of the line and increase the pressure at this point, thus interfering with the flow.

When possible a pipe line should be laid so as not to rise above its hydraulic gradient. The pressure throughout the line is thus maintained above atmospheric and air valves placed at all summits allow the escape of air which tends to accumulate at these points.

If the formation of the ground makes it necessary to lay the pipe line above its hydraulic gradient the pipe line becomes a siphon. Pumps for the removal of air must then be installed at the summit in order to maintain the flow. Under such conditions, Fig. 65, the pressure at the summit is evidently h_1 feet of water below atmospheric pressure; h_1 can never be more than 34 feet and practically it is limited to 25 feet, if a flow of water is to occur.

If the pipe line in Fig. 65 is open to the atmosphere at the summit or if sufficient leakage occurs to increase the pressure at the summit to the atmospheric pressure, the hydraulic gradient becomes *AB* and the available head at the summit is reduced by h_1 feet. The velocity and the quantity of water flowing per second is thus reduced and the portion of the pipe *BC* no longer flows full. A smaller pipe from *B* to *C* would thus serve to carry away the water delivered by the larger pipe at *B*.

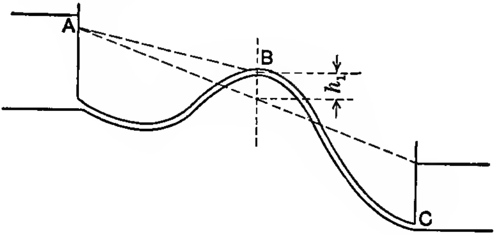

FIG. 65.

EXERCISE 135. A pipe line 2000 feet long and 10 inches in diameter connects two reservoirs. The difference in level of the free surfaces is 30 feet. The line rises 10 feet above the free surface in the higher reservoir. The length of the pipe between the higher reservoir and the summit is 500 feet. The pipe line discharges 5 feet below the free surface of the lower reservoir. Assume $f=0.0053$ and compute

(*a*) the velocity of flow in the pipe,

(*b*) the pressure at the summit.

(*c*) If the absolute pressure maintained at the summit is 20 feet of water find the quantity of water transferred to the lower reservoir per second.

SECTION XVIII

SECONDARY LOSSES IN PIPE LINES

Losses Due to Sudden Enlargement of Section.—As illustrated in Fig. 66 let v_1 and v_2 be the velocities in the pipes whose sectional areas are A_1 and A_2. Also let $(p_1 - p_a)$ and $(p_2 - p_a)$ be the corresponding pressures as indicated by the piezometer tubes.

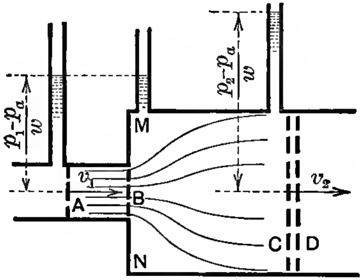

FIG. 66.

Experiment shows that the pressure in the enlarged portion of the pipe does not at once rise to p_2, but that at *M* and *N* the pressure in the eddying water remains equal to p_1. Note the mechanical explanation for this in the curvilinear path of the particles of water while adjusting themselves to the new velocity v_2 (see page 75).

Consider now the displacement of a section from *A* to

B and the simultaneous displacement of a section from C to D during the time Δt. The condition of the liquid between sections B and C remains unchanged even though a part of the liquid itself has been replaced by the liquid originally between sections A and B. Thus the mass of liquid retarded during the passage of the enlargement is the mass contained between sections C and D or its equivalent between A and B. This mass equals

$$\frac{A_2 v_2 \Delta t\, w}{g}.$$

The change in velocity is $v_2 - v_1$ and the acceleration

$$\frac{v_2 - v_1}{\Delta t}.$$

The force causing this acceleration is

$$A_1 p_1 + (A_2 - A_1) p_1 - A_2 p_2 = A_2 p_1 - A_2 p_2.$$

Therefore

$$A_2(p_1 - p_2) = \left(\frac{A_2 v_2 \Delta t\, w}{g}\right)\left(\frac{v_2 - v_1}{\Delta t}\right),$$

whence $$\frac{p_2 - p_1}{w} = \frac{v_2(v_1 - v_2)}{g}.$$

Now Bernoulli's theorem states that:

$$\frac{p_1}{w} + \frac{v_1^2}{2g} = \frac{p_2}{w} + \frac{v_2^2}{2g} + h_l,$$

where h_l is the loss (of head or of energy per pound) due to enlargement of the section.

Thus $$h_l = \frac{p_1 - p_2}{w} + \frac{v_1^2 - v_2^2}{2g},$$

or

$$h_l = -\frac{v_2(v_1 - v_2)}{g} + \frac{v_1^2 - v_2^2}{2g}$$

$$= \frac{v_2^2 - 2v_2\,v_1 + v_1^2}{2g}$$

$$= \frac{(v_2 - v_1)^2}{2g} \text{ foot-pounds per pound of water.}$$

Exercise 136. A pipe suddenly enlarges from 3″ to 4″ in diameter. 100 gallons of water flow from the 3″ portion to the 4″ portion per minute. What head is lost at the enlargement?

Exercise 137. Show that the loss due to the sudden enlargement of a pipe from a diameter of D_1 feet to a diameter of D_2 feet is

$$\left(\frac{D_2^2}{D_1^2} - 1\right)^2 \frac{v_2^2}{2g},$$

where v_2 is the velocity in the pipe of larger diameter.

In order to avoid this loss of head due to sudden enlargement of the section the change in section should be gradual. A batter of 1 to 10 with velocities less than 6 feet per second practically eliminates this loss of head.

When a **pipe discharges into a reservoir** below the surface of the water the loss due to eddies may be computed by means of the formula $\frac{(v_2 - v_1)^2}{2g}$. Under these conditions v_2 becomes zero, so that the head lost during discharge would be $\frac{v_1^2}{2g}$.

Loss Due to Sudden Contraction of Section.—Fig. 67 shows the probable stream lines for this case. The loss of head may be conceived as due to two causes:

1. The contraction from section 1 to section 2 which is similar in nature to the loss at an orifice, and

2. The loss due to enlargement of section in passing from section 2 to section 3.

Thus the total head lost is

$$h_l = \left(\frac{1}{c_v^2} - 1\right)\frac{v_2^2}{2g} + \frac{(v_2 - v_3)^2}{2g}.$$

If $c_c = \frac{A_2}{A_3}$ is the coefficient of contraction, then by means of the equation of continuity, $A_2v_2 = A_3v_3$, we find that

$$v_2 = \frac{v_3}{c_c},$$

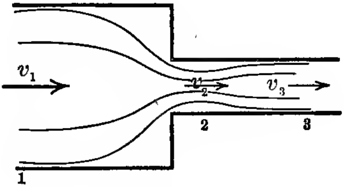

FIG. 67.

and
$$h_l = \left(\frac{1}{c_v^2} - 1\right)\frac{1}{c_c^2}\,\frac{v_3^2}{2g} + \left(\frac{v_2}{v_3} - 1\right)^2 \frac{v_3^2}{2g}$$

$$= \left[\left(\frac{1}{c_v^2} - 1\right)\frac{1}{c_c^2} + \left(\frac{1}{c_c} - 1\right)^2\right]\frac{v_3^2}{2g}.$$

The constants, c_v and c_c, upon which the value of this bracketed coefficient depends cannot be determined with any degree of accuracy. This coefficient is thus determined experimentally. For values of this coefficient the student is referred to his engineering hand-books. As a special case may be noted the loss at entrance to a pipe (see page 110).

Losses Due to Elbows, Bends, Valves, and Gates.—No theory has been devised which leads to even approximate values for these losses. Experimental investigations of these losses are few in number and they were usually made on pipes of small diameter. Fortunately on long pipe lines the frictional resistances are so much larger than the secondary resistances that these need seldom be taken into account. For the empirical coefficients by which $\frac{v^2}{2g}$ should be multiplied in order to obtain the loss due to these secondary causes the student is referred to engineering hand-books and treatises on hydraulics.

SECTION XIX

FLUID FRICTION

The Frictional Force Exerted by Liquids upon Plane and Cylindrical Surfaces.—The surface of a solid dragged through a liquid is acted upon by a frictional force tending to retard the motion. As motion is relative the frictional force exerted by a liquid moving over a stationary solid equals the frictional force exerted by the same solid dragged through the liquid at rest provided the relative velocities are the same.

Consider the horizontal pipe, Fig. 68, through which a liquid is flowing with a velocity v.

Between sections (1) and (2) there is a loss of head $h_f=\frac{p_1}{w}-\frac{p_2}{w}$ due to frictional resistance. The loss of head equals $f\frac{4l}{D}\,\frac{v^2}{2g}$, where f is the friction factor and D the diameter of the pipe.

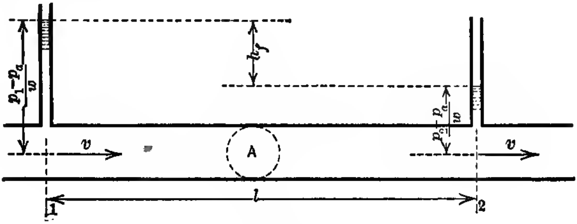

FIG. 68.

If we make a free body of the liquid between sections (1) and (2), Fig. 69, and represent the sum of the horizontal frictional forces exerted by the pipe upon the liquid by R, we see that the equation $F=ma$ gives

$$p_1\,A-R-p_2\,A=0,$$

or

$$R=(p_1-p_2)\,A.$$

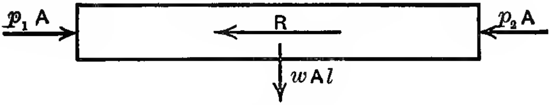

FIG. 69.

But

$$(p_1-p_2)=wh_f=w\,f\,\frac{4l}{D}\,\frac{v^2}{2g},$$

so that

$$R=wf\,\frac{4l}{D}\,\frac{v^2}{2g}\,\frac{\pi D^2}{4}=wf(\pi\,Dl)\frac{v^2}{2g},$$

and as $\pi Dl =$ the area of the surface of contact between pipe and liquid we may state that:

The frictional force exerted by a moving liquid upon the surface of a solid in contact with it equals the continued product of the weight of a cubic foot of the liquid, the friction factor, the area of the surface, and $\frac{v^2}{2g}$.

EXERCISE 138. Show by means of the assumptions and constants given on page 112 that the frictional resistance per unit area of the surface of contact equals $wf\frac{v^2}{2g}$.

It should be noted that not only is the liquid held back by the force R, but the pipe is dragged forward by an equal force R, and thus tends to move with the liquid.

EXERCISE 139. A disc, diameter d, of negligible thickness is rotated at a constant angular velocity ω in water. Compute the torque exerted upon the disc by a shaft of negligible diameter, the friction factor being assumed constant for all velocities.

The Webb Viscous Dynamometer.—In this dynamometer several thin steel discs are fastened to the shaft to which the power to be measured is transmitted. These discs are surrounded by a cylindrical case to the inside of which are fastened annular discs which lie between the discs fastened to the shaft. The case is supported so as to be free to rotate about the same geometrical axis as the shaft but is in no way connected to it. Water is supplied to and removed from the case in a continuous stream. This water is heated by the energy absorbed by the brake. This energy is of course wasted.

When the shaft and its discs are rotating the water fills that part of the case furthest removed from the shaft.

Friction imparts motion to the water and it in turn imparts motion to the case. The tendency of the case to move with the shaft is prevented by a band passing about the case and fastened to a spring balance. The torque exerted on the case can thus be measured. It is equal to the torque exerted upon the water by the discs attached to the shaft and also equals the resisting torque exerted on the shaft.

EXERCISE 140. In a Webb Viscous Dynamometer
let N, be the number of discs fastened to the shaft;
r_2, the extreme radius of these discs;
r_1, the inner radius of their wetted surfaces;
f, the friction factor;
n, the number of revolutions per second of the shaft.
Show that if f is assumed constant the horse power absorbed equals

$$1.098\, f\, N\, n^3(r_2^5 - r_1^5).$$

EXERCISE 141. Experiment shows that one plate 24 inches in diameter completely wetted absorbs 100 horse-power when rotating at 2400 rev. per minute. Find the friction factor.

EXERCISE 142. A Webb Viscous Dynamometer having ten plates 16 inches in diameter wetted to within 4 inches of the center absorbs 30 H.P. when running at 1500 revolutions per minute. Compute f.

CHAPTER VII

FORCES EXERTED BY MOVING LIQUIDS

SECTION XX

FORCES EXERTED ON PIPES

LIQUIDS exert forces due to their motion when their velocity is changed, either in magnitude or direction. In addition to the forces now to be considered liquids of course always exert forces due to the static pressure and due to the frictional resistances encountered. The forces due to change of velocity are sometimes called dynamic forces to distinguish them from the static forces due to pressure.

The fundamental equation of mechanics involving changes in velocity is

$$\text{force} = \text{mass} \times \text{acceleration}.$$

The acceleration may be due to changes in the magnitude of the velocity as in rectilinear motion with varying velocity, or it may be due to changes in the direction of a constant velocity as in circular motion with constant velocity, or it may be due to changes in magnitude and in direction occurring simultaneously.

Forces Due to Change in Magnitude of Velocity.—In Fig. 70 is shown a pipe whose sectional area diminishes gradually from A_1 to A_2. Assume the pressure and the velocity of the liquid to be p_1 and v_1 at (1) and p_2 and v_2 at (2), then

$$A_1v_1=A_2v_2 \quad \text{and} \quad \frac{p_1}{w}+\frac{v_1^2}{2g}=\frac{p_2}{w}+\frac{v_2^2}{2g},$$

provided no energy is gained or lost during the flow from (1) to (2).

Consider now the liquid between sections AB and CD as a free body. The force acting upon this mass of liquid must equal its mass times the acceleration produced. Let the displacement considered occur in Δt seconds during which time the section AB moves to $A'B'$ and the section CD moves to $C'D'$, evidently the distance $AA'=v_1\,\Delta t$ and $CC'=v_2\,\Delta t$. During this motion the mass of liquid originally between sections $A'B'$ and CD has been replaced by an equal mass which is assumed to be particle for particle under the same conditions as the liquid which originally occupied this position. Thus the only mass which really suffers change is the mass between sections AB and $A'B'$ which may be conceived to be transferred to $CC'D'D$.

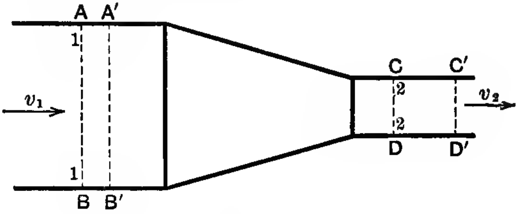

FIG. 70.

This mass is $$A_1v_1\,\Delta t\,\frac{w}{g},$$

and its velocity is changed from v_1 to v_2 in Δt seconds, so that the acceleration is $\frac{v_2-v_1}{\Delta t}$.

The force required to produce this change is

$$F=\left(A_1 v_1 \Delta t \frac{w}{g}\right)\left(\frac{v_2-v_1}{\Delta t}\right)$$

$$=\frac{A_1 v_1 w}{g}(v_2-v_1).$$

Therefore during the flow considered in Fig. 70 a **constant force in the direction of the axis of the pipe equal to the mass of liquid passing any section of the pipe per second multiplied by the change in velocity** between the two sections considered must have acted upon the liquid between these sections.

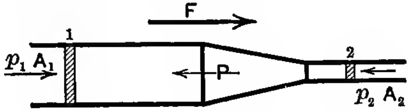

FIG. 71.

This force F suffers no displacement, for no matter how long the flow continues the conditions between sections AB and CD remain unchanged. Moreover this force F does no work, for no energy is created by reason of the flow considered.

To find the force exerted by the pipe upon the water we may proceed as follows:

In Fig. 71 the conditions of Fig. 70 are reproduced by means of a piston (1) moving with a velocity v_1 and acted upon by the force p_1A_1, and a piston (2) moving with a velocity v_2 and acted upon by a force p_2A_2 as shown.

The total horizontal force acting upon the liquid towards the right is $\frac{A_1 v_1 w}{g}(v_2-v_1)$, as above computed.

If P represents the horizontal component of the force (acting towards the left) exerted by the pipe upon the liquid

we have, $$p_1A_1-P-p_2A_2=\frac{A_1v_1w}{g}(v_2-v_1),$$

or $$P=p_1A_1-p_2A_2-\frac{A_1v_1w}{g}(v_2-v_1).$$

The following computation of the force P will impress upon the reader the simplicity of the method used above.

The only portion of the pipe which can exert a horizontal force upon the contained liquid is the conical portion.

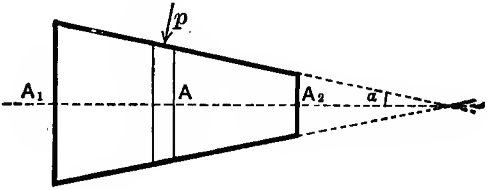

FIG. 72.

Direct computation of this horizontal force may be made as follows.

In Fig. 72, the pressure p at any section A may be obtained from

$$\frac{p_1}{w}+\frac{v_1^2}{2g}=\frac{p}{w}+\frac{v^2}{2g},$$

where $$v=\frac{A_1}{A}v_1.$$

The band upon which p acts has an area of $\frac{dA}{\sin\alpha}$. The pressure p is everywhere normal to this surface and it is inclined at an angle α to the vertical. The horizontal component of the resulting force acting upon this differential band is

$$p\frac{dA}{\sin\alpha}\,\sin\alpha = p\,dA.$$

The resultant horizontal force equals

$$\begin{aligned}
P &= \int_{A_2}^{A_1} p\,dA = w\int_{A_2}^{A_1}\left(\frac{p_1}{w}+\frac{v_1^2}{2g}-\frac{v_1^2}{2g}\,\frac{A_1^2}{A^2}\right)dA \\
&= w\left[\frac{p_1}{w}(A_1-A_2)+\frac{v_1^2}{2g}(A_1-A_2)+\frac{v_1^2}{2g}A_1^2\left(\frac{A_2-A_1}{A_1A_2}\right)\right] \\
&= p_1(A_1-A_2)+w\frac{v_1^2}{2g}\left[1-\frac{A_1}{A_2}\right](A_1-A_2) \\
&= p_1(A_1-A_2)+w\frac{v_1^2}{2g}\left(1-\frac{v_2}{v_1}\right)\left(\frac{v_2}{v_1}-1\right)A_2 \\
&= p_1A_1-p_1A_2+\frac{w\,A_2}{2g}(2\,v_1\,v_2-v_1^2-v_2^2) \\
&= p_1A_1-w\left(\frac{p_2}{w}+\frac{v_2^2}{2g}-\frac{v_1^2}{2g}\right)A_2+\frac{w\,A_2}{2g}(2v_1v_2-v_1^2-v_2^2) \\
&= p_1A_1-p_2A_2-\frac{w\,A_2}{2g}\{v_2^2-v_1^2-2v_1v_2+v_1^2+v_2^2\} \\
&= p_1A_1-p_2A_2-\frac{w\,A_2v_2}{g}(v_2-v_1),
\end{aligned}$$

which agrees with the value of P deduced above by means of the shorter method.

Thus the flowing water exerts a force P upon the conical portion of the pipe. This force acts towards the right, and unless the foundations are capable of resisting this force the pipe will move towards the right. It should be noted that this force does not include the frictional effect discussed on page 125.

EXERCISE 143. (*a*) If the piston at (2) Fig. 71 is fastened to the pipe, compute the force exerted upon the liquid by the pipe under the action of a force p_1A_1 acting on the piston at (1).

(*b*) Compute the force exerted upon this pipe if the liquid moves towards the left.

Forces Due to Change in Direction of Velocity.—The force exerted by moving liquids due to a change in the direction of the flow can best be illustrated by means of a bend in a pipe of constant diameter.

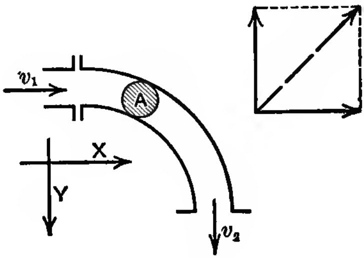

FIG. 73.

Consider a pipe bend (Fig. 73), lying in a horizontal plane. Water enters it with a velocity v_1 and is discharged into the atmosphere at right angles to the original direction of flow with a velocity v_2. Losses neglected, the pressure of the liquid throughout will be atmospheric and the magnitude of the velocity v_2 will equal the magnitude of the velocity v_1.

Due to its passage through the bend the velocity v_1 of the liquid in the X-direction is completely destroyed. Therefore in accordance with the fundamental theorem that the force exerted upon a liquid in any direction in order to change its velocity in that direction must equal the mass of liquid flowing per second multiplied by the change in the velocity in the direction considered, the force exerted upon the liquid in the X-direction equals

$$M(0-v_1)=-M\,v_1=-\frac{A\,v_1w}{g}v_1.$$

This force can be exerted only by the pipe, so that the pipe in turn must be pushed by the flowing liquid in the X-direction with a force

$$\frac{A\,v_1w}{g}v_1.$$

Similarly in the Y direction the force acting on the liquid is

$$\frac{A\,v_1w}{g}(v_2-0)=\frac{A\,v_1w}{g}v_2,$$

and on the pipe,

$$-\frac{A\,v_1w}{g}v_2.$$

As $v_1=v_2$, the resultant force on the pipe bend is

$$\sqrt{2\left(\frac{A\,v_1^2w}{g}\right)^2}=\frac{A\,v_1^2w}{g}\sqrt{2}.$$

Instead of computing the X- and the Y-components of the forces in order to obtain the resultant force we may proceed as follows:

The initial velocity of the water is v_1 (Fig. 74) the final velocity is v_2. The change in velocity is **always** found by subtracting the initial from the final velocity. To perform this subtraction graphically reverse the direction of the initial velocity, obtaining v_1', and then add this velocity to the final velocity v_2, obtaining the change in velocity $v' = v_1\sqrt{2}$.

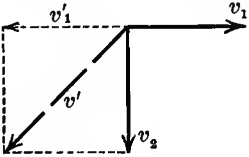

FIG. 74.

The resultant force exerted by the pipe on the water

$$= \text{(mass per second) (change in velocity)}$$

$$= \frac{A v_1 w}{g}(v_1\sqrt{2}) = \frac{A\, w\, v_1^2}{g}\sqrt{2},$$

and this force acts in the direction of the arrow v', Fig. 74.

EXERCISE 144. Compute the force exerted by a liquid flowing with a velocity v through the bend shown in Fig. 75, the pressure throughout being atmospheric.

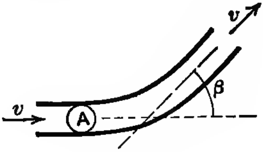

FIG. 75.

If the pressure in the pipe is not atmospheric the statical forces exerted by the liquid must be added to the dynamic forces.

As the statical force exerted is normal to the differential surface on which the pressure acts, the component of this force in any direction equals

$$(dF)\ (\cos \alpha) = (p\ dA)\ (\cos \alpha)$$

where α is the angle which the normal to the surface dA makes with the direction considered. This component of the force also equals

$$p\ (\cos \alpha\ dA)$$

where $\cos \alpha\, dA$ is the projection of the area of the differential surface upon a plane normal to the direction considered. Therefore, the statical force upon any surface equals the pressure multiplied by the area of the projection of the surface upon a plane normal to the direction in which the statical force is required.

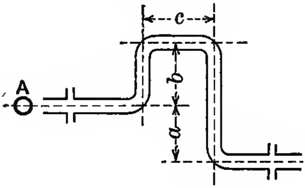

FIG. 76.

EXERCISE 145. If a hemispherical cover is bolted to the open end of the bend shown in Fig. 73, compute the force exerted on the bend by a liquid filling the bend under a pressure p.

EXERCISE 146. Compute the resultant force upon the bent pipe, Fig. 76, when a liquid is flowing through it with a velocity v under a pressure p.

SECTION XXI

FORCES EXERTED ON STATIONARY DEFLECTING SURFACES

Let a jet, moving with a velocity of v feet per second, be deflected through an angle β by a fixed surface as shown in Fig. 77. Neglecting the loss of velocity due to friction, which will be small owing to the small surface of contact between the liquid and solid, the magnitudes of the velocities v_1 and v_2 at entrance and at exit will be equal to each other and also equal to v.

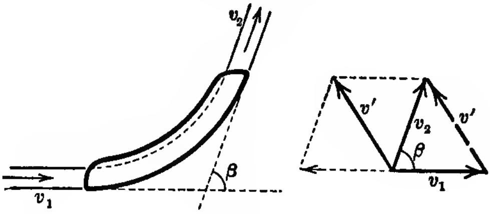

FIG. 77.

The change in the velocity, v', due to the deflection of the jet is found by subtracting the initial velocity v_1 from the final velocity v_2. This subtraction may be performed by reversing v_1 and adding it to v_2 as shown by the parallelogram construction in Fig. 77.

A shorter process for obtaining v' is also indicated in Fig. 77. Simply join the end of the initial velocity v_1 with the end of the final velocity v_2, after they have been placed tail to tail, in order to obtain v'. Note the direction of this arrow, from the head of the initial velocity to the head of the final velocity.

Analytically we find that

$$v'^2 = v_1^2 + v_2^2 - 2v_1v_2 \cos \beta$$

and as

$$v_1 = v_2 = v,$$

$$v' = v\sqrt{2(1 - \cos \beta)}.$$

If M is the mass of the liquid reaching the surface per second, the resultant force exerted upon the liquid by the surface equals

$$F = Mv' = Mv\sqrt{2(1 - \cos \beta)},$$

and this resultant has the direction of v' shown in Fig. 77.

The force exerted upon the surface is of course equal to F and acts in a direction opposed to the arrows v'.

EXERCISE 147. A nozzle having an area of 3 square inches discharges water under an effective head of 100 feet at the base of the nozzle. There is no contraction of the jet and the coefficient of velocity of the nozzle is 0.90. The jet enters a fixed, smooth, semicircular channel without shock and is deflected by it. What force does the jet exert upon this channel?

EXERCISE 148. How much kinetic energy does the stream in Exercise 147 lose? How can the force exerted be accounted for?

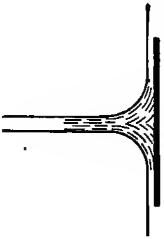

FIG. 78.

When **a jet strikes a flat surface normally** as in Fig. 78, the velocity of the jet in its original direction is destroyed and as an approximation it may be assumed that the liquid flows parallel to the plane in all directions.

EXERCISE 149. A pipe 2 feet in diameter and 3000 feet long delivers water from a reservoir to a nozzle placed 1500 feet below the free surface of the reservoir. What will be the maximum pressure exerted upon a vertical cliff by the horizontal jet produced in this manner if the area over which the pressure is assumed to be uniformly distributed equals four times the sectional area of the jet? (See Exercises 133 and 134.)

The Reaction of a Jet.—Assume water to flow through an orifice (area a) in the side of a tank (Fig. 79) with a velocity v without contraction or loss of velocity through friction. The horizontal velocity of the flowing water is changed from zero at the free surface to v at the orifice

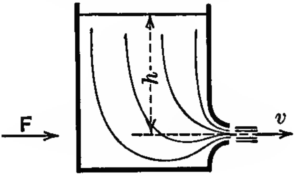

FIG. 79.

and the horizontal force exerted upon the water leaving the orifice must be

$$F=\frac{w\,a\,v}{g}\,(v),$$

directed towards the right. Therefore an equal force F directed towards the left must react upon the tank, which is the only body in contact with, and therefore capable of acting upon, the water.

To maintain equilibrium a force equal to F must act upon the tank in the direction of v.

The tank thus acts as a deflecting surface.

The force F may also be computed by means of the principle of work as follows:

Assume the force F to act as shown in Fig. 79 and that the tank moves towards the left with a constant velocity, u, under the action of this force and no other horizontal force excepting such as may be exerted by the water.

In a time dt the weight of water discharged would be $w\,a\,v\,dt$ and this weight of water may be assumed to have passed from the free surface through the orifice. The absolute velocity of the water at the free surface was u, neglecting the velocity of approach. The absolute velocity on leaving the tank is $(v-u)$, where $v=\sqrt{2\,gh}$ is the velocity of the issuing water relative to the tank.

The gain in kinetic energy is thus

$$\left(\frac{w\,a\,v\,dt}{g}\right)\left\{\frac{(v-u)^2}{2}-\frac{u^2}{2}\right\}=\frac{w\,a\,v\,dt}{g}\left(\frac{v^2-2\,uv}{2}\right).$$

This gain in kinetic energy must equal the work done by all the forces acting upon the water. These forces are the force due to gravitation and the force, F, exerted by the tank. The work done by these forces amounts to

$$(w\,a\,v\,dt)\,h-F\,(udt).$$

Thus by the principle of work,

$$(w\,a\,v\,dt)\,h-F\,(udt)=\frac{w\,a\,v\,dt}{g}\left(\frac{v^2-2\,uv}{2}\right),$$

whence $$Fu=(wav)\left(h-\frac{v^2-2\,uv}{2g}\right),$$

but $$h=\frac{v^2}{2g},$$

so that $$Fu=\frac{w\,a\,v}{2g}(2\,u\,v),$$

or $$F=\frac{w\,a\,v^2}{g}.$$

EXERCISE 150. Show that this force F equals twice the static force on a stopper closing the orifice. Explain why F should be greater than the static force.

EXERCISE 151. A cubic foot of water issues per second with a velocity v from an orifice in a tank, Fig. 80.

(*a*) What force acts upon the tank and in what direction does it act?

(*b*) What force acts upon the plate AB?

(*c*) Do these results imply that the jet is in compression?

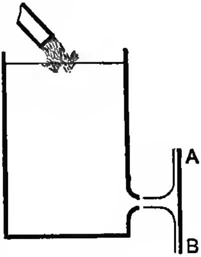

FIG. 80.

Jet Propulsion.—It has often been proposed to propel ships by means of the reaction of jets of water. As illustrated in Fig. 81 consider a ship moving with an absolute velocity u. Water is taken in at the bow and after passing through the pump it is expelled with a velocity v **relative to the ship.** Under ideal conditions the water would enter the ship with a relative velocity u; it has then no absolute velocity. The pumps thus change the velocity of the water from u to v relative to the ship or by $v-u$. If the area of the delivery pipe is a the pump must exert a force of

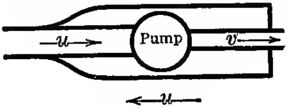

FIG. 81.

$$\frac{w\,a\,v}{g}(v-u)$$

upon the water. The water reacts upon the pump and thus upon the ship with the same force in the forward direction.

The work performed is

$$\frac{w\,a\,v(v-u)}{g}u \text{ foot-pounds per second.}$$

EXERCISE 152. Compute the power required at the pump under ideal conditions and show that the efficiency of propulsion is

$$\frac{2\,u}{v+u}.$$

SECTION XXII

FORCES EXERTED ON AND ENERGY TRANSMITTED TO TRANSLATING DEFLECTING SURFACES

Assume the surface deflecting the jet constrained to move in the direction u with a constant velocity u (Fig. 82).

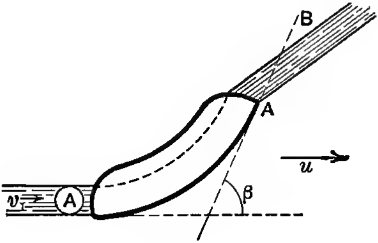

FIG. 82.

Thus sufficient horizontal force must be applied to the deflecting surface towards the left to just balance the horizontal force exerted by the jet towards the right; otherwise accelerated motion would result. Under these conditions work is done by the water and this work should equal the loss of kinetic energy of the water.

The relative velocity with which the water reaches the moving vane is $v_1 - u$. Neglecting friction this relative velocity remains constant in magnitude but changes its direction. The relative velocity of the jet at entrance to the vane is obtained by subtracting the velocity of the vane from the velocity of the jet, Fig. 83 (*a*), there results ${}_rv_1 = v_1 - u$. The jet leaves the vane in the direction AB (Fig. 82) relative to the vane, but the vane actually has a horizontal velocity so that the absolute velocity of exit, v_2, is the sum of ${}_rv_1 (= v_1 - u)$ in the direction AB (Fig. 82) and the horizontal velocity u. v_2 can be found as shown in Fig. 83 (*b*).

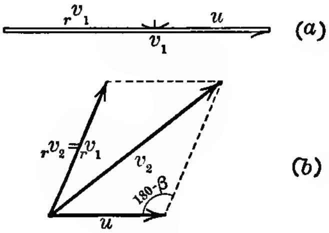

FIG. 83.

Analytically $\qquad v_2^2 = u^2 + {}_rv_1^2 - 2\,u\,{}_rv_1 \cos(180 - \beta)$

$$= u^2 + {}_rv_1^2 + 2\,u\,{}_rv_1 \cos \beta.$$

The force exerted on the vane equals the mass of water reaching the vane per second multiplied by the change in the relative velocity.

The mass reaching the vane per second is

$$\frac{A\ w\ {}_rv_1}{g}=\frac{A\ w\ (v_1-u)}{g}.$$

The change in the relative velocity is found as shown in Fig. 84.

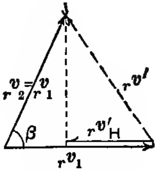

FIG. 84.

Analytically

$$\begin{aligned}{}_rv'^2&={}_rv_1^2+{}_rv_2^2-2\ {}_rv_1\ {}_rv_2\cos\beta\\&=2\ {}_rv_1^2-2\ {}_rv_1^2\cos\beta\\&=2\ (1-\cos\beta)\ {}_rv_1^2.\end{aligned}$$

Therefore the total force exerted upon the vane is

$$\begin{aligned}F&=\frac{A\ w(v_1-u)}{g}\sqrt{2(1-\cos\beta)}\ (v_1-u)\\&=\frac{A\ w(v_1-u)^2}{g}\sqrt{2(1-\cos\beta)}.\end{aligned}$$

Of this force only the horizontal component can do work. This component force can most readily be found by multiplying the mass of water reaching the vane per second by the horizontal component of the change in the relative velocity, which from Fig. 84, equals

$${}_rv_H'={}_rv_1-{}_rv_1\cos\beta=(v_1-u)\ (1-\cos\beta).$$

The horizontal force acting upon the vane is

$$F_H=\frac{A\ w(v_1-u)}{g}(v_1-u)\ (1-\cos\beta)=\frac{A\ w(v_1-u)^2}{g}(1-\cos\beta).$$

The work this force does per second is

$$F_H\,u=\frac{A\ w(v_1-u)^2u}{g}(1-\cos\beta).$$

This work should equal the loss in kinetic energy of the jet. The kinetic energy supplied by the jet per second depends upon its absolute velocity and the mass of water supplied per second.

Thus the power of the jet reaching the vane was

$$\frac{A\ w\ (v_1-u)\ v_1^2}{2g}\ \text{foot-pounds per sec.}$$

The power of the jet leaving the vane is

$$\frac{A\ w(v_1-u)}{g}\ \frac{v_2^2}{2}\ \text{foot-pounds per sec.}$$

So that the jet has lost in power

$$\frac{A\ w(v_1-u)}{2g}(v_1^2-v_2^2)=\frac{A\ w(v_1-u)}{2g}\left\{v_1^2-(u^2+{}_rv_1^2+2u\,{}_rv_1\cos\beta)\right\}$$

$$=\frac{A\ w(v_1-u)}{2g}\left\{v_1^2-(u^2+v_1^2-2\,u\,v_1+u^2+2\,u\,v_1\cos\beta-2\,u^2\cos\beta)\right\}$$

$$=\frac{A\ w(v_1-u)}{2g}(2\,u\,v_1-2u^2-2uv_1\cos\beta+2u^2\cos\beta)$$

$$=\frac{A\ w(v_1-u)}{2g}\left\{2\,u\,v_1(1-\cos\beta)-2\,u^2(1-\cos\beta)\right\}$$

$$=\frac{A\ w(v_1-u)}{g}u(v_1-u)\ (1-\cos\beta)$$

$$=\frac{A\ w(v_1-u)^2\,u}{g}(1-\cos\beta).$$

CHAPTER VIII

AXIAL-FLOW IMPULSE WHEELS

SECTION XXIII

TANGENTIAL WATER WHEELS

IN an impulse water wheel the pressure of the moving water is reduced to atmospheric pressure before the water reaches the moving vanes. This reduction of the pressure with corresponding increase in velocity is effected by means of nozzles or fixed guide vanes.

During the passage of the water over the moving vanes its pressure remains constantly equal to the atmospheric pressure.

The whole of the potential energy, whether it is due to position or to pressure, is thus first converted into kinetic energy before the water enters the moving vanes, which are designed so as to extract as much of this kinetic energy from the moving water as possible.

In an axial-flow impulse water wheel the water while passing over the moving vanes flows in a plane determined by a line tangent to the periphery of the wheel and a line parallel to the axis of the wheel. The water neither moves towards nor away from the axis.

The vanes are attached to the periphery of the wheel and thus revolve about the axis of the wheel. Any given particle of water is, however, in contact with a vane for so short a time that during this time the motion of the vane may be assumed, as a first approximation, to be one of translation with a velocity equal to the peripheral velocity of the wheel.

Conceive the vane shown in Fig. 85 to be bounded by cylindrical surfaces (whose elements are perpendicular to the plane of the paper) and attached to the periphery of a wheel whose plane is perpendicular to the paper and whose axis is *AB*.

The jet (cross-section *A*) moving with an absolute velocity v_1 is parted by the vane and one-half of the water is thrown to either side of the plane of the wheel.

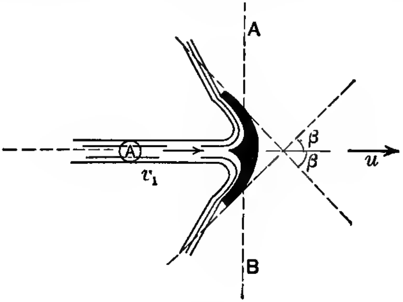

FIG. 85.

The force exerted on the vane, the work done by this force, and the loss of kinetic energy of the water may be computed by the methods explained on the last pages of the last section.

EXERCISE 153. Compute the force exerted by the jet upon each half of the vane shown in Fig. 85, assume the vane to be moving with a velocity *u*. Note the direction of these forces.

EXERCISE 154. Compute the effective driving force exerted on the vane shown in Fig. 85.

A single vane acted upon by a jet of water has no practical application, for the vane soon passes out of the action of the jet. A series of vanes must be attached to the periphery of the wheel so that the vanes going out of action may be replaced by other vanes, the action being thus made continuous.

The number of vanes in action at any time may be found as follows:

In Fig. 86, let x be the distance between successive vanes

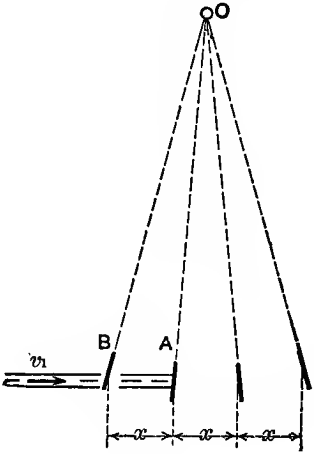

FIG. 86.

revolving about the axis O but assumed to translate with a velocity u in the direction of the jet. Then vane A has been in action

$$\frac{x}{u} \text{ seconds}$$

when vane B comes into action.

The portion of the jet between A and B (now cut off from the jet by vane B) continues to travel with an absolute velocity v_1 while vane A moves with an absolute velocity u. To find the time during which this portion of the jet continues to act upon vane A note that its length is x and that it moves with a relative velocity v_1-u with respect to the vane A. Therefore some portion of this part of the jet will remain in contact with vane A during

$$\frac{x}{v_1-u} \text{ seconds.}$$

The elapsed time from the moment vane A comes into action until it goes out of action is thus

$$\frac{x}{u}+\frac{x}{v_1-u}=\frac{v_1 x}{u(v_1-u)} \text{ seconds.}$$

During this time vane A moves

$$\left\{\frac{v_1 x}{u(v_1-u)}\right\}(u)=\frac{v_1 x}{v_1-u} \text{ feet.}$$

The number of vanes that are in action at any time is therefore

$$\frac{\dfrac{v_1 x}{v_1-u}}{x}=\frac{v_1}{v_1-u}.$$

Exercise 155. (*a*) By means of the result of Exercise 154 find the total driving force exerted on the wheel described above.

(*b*) Compute the power transmitted to this wheel.

Note that a series of vanes are acted upon by the whole mass of water supplied per second.

Exercise 156. Compute the absolute velocity of exit from the vane shown in Fig. 85.

Exercise 157. If Q is the quantity of water discharged by a nozzle per second with a velocity v_1, compute the kinetic energy lost by this water during deflection by the vanes shown in Fig. 85; and show that this result equals the result in Exercise 155 (*b*).

The ideal efficiency of a series of vanes is evidently equal to the power transmitted to the wheel (or the loss of kinetic energy of the water per second) divided by the initial kinetic energy of the jet.

Exercise 158. (*a*) Compute the ideal efficiency of the vanes shown in Fig. 85.

(*b*) With a given vane and a given jet velocity for what value of u is the efficiency a maximum?

(*c*) What is the maximum efficiency when β, Fig. 85, equals (*a*) 15°, (*b*) 30°?

Exercise 159. (*a*) Compute the absolute exit velocity when $\beta = 0°$, and the wheel runs at maximum efficiency.

(*b*) Why cannot β be made equal to 0°?

Tangential water wheels are axial impulse wheels fitted with patented buckets * shaped so as to disturb the jet as little as possible when they enter into action and so as to maintain axial flow during the time that the bucket is in action. These buckets are similar to the vanes shown in Fig. 85, β being made as small as is consistent with clearing the wheel of the spent water.

Exercise 160. In Pelton water wheel practice buckets having an actual efficiency of at least 80% have been developed for nozzles having a diameter not greater than 1/18 of the effective diameter of the wheel.

How many nozzles (jets), n, must be used to develop H horse-power with a wheel whose radius is R feet and with an available head of h feet? Assume the efficiency to be 80%, the coefficient of contraction 0.97, and the wheel running at the speed corresponding to maximum efficiency.

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* For illustrations see *Scientific American*, Oct. 5, 1912, and Catalogues of the Pelton and the Doble Company.

Exercise 161. (*a*) If it is impracticable to play more than three jets on one wheel, how many tangential wheels 21 inches in diameter are required to generate 400 horse-power with 350 feet of head?

(*b*) How many revolutions per minute should these wheels make?

(*c*) What should be the diameter of each nozzle?

(*d*) If the pipe line supplying these wheels is 5000 feet long, compute the best diameter for the pipe leading to the nozzles (see Exercise 131).

Exercise 162. It is proposed to run an air compressor having a flywheel 10 feet in diameter and absorbing 150 horse-power by means of Doble buckets attached to the rim of the flywheel. The available head is 300 feet.

(*a*) How many cubic feet of water are required per second, assuming an efficiency of 80%?

(*b*) What total nozzle area is required ($c_c = 0.97$)?

(*c*) What must be the diameter of the nozzle?

(*d*) How many revolutions must the compressor make per minute?

Section XXIV

GENERAL CASE OF TRANSLATING VANES DEFLECTING A JET

Let W be the weight of water striking the vanes per second;

v_1, the absolute velocity of the water at entrance to the vanes;

v_2, the absolute velocity of the water at exit of the vanes;

u, the absolute velocity of translation of the vanes;

$_rv_1$, the relative velocity of the water with respect to the vanes at entrance;

$_rv_2$, the relative velocity of the water with respect to the vanes at exit;

$_rv'$, the change in the relative velocity;

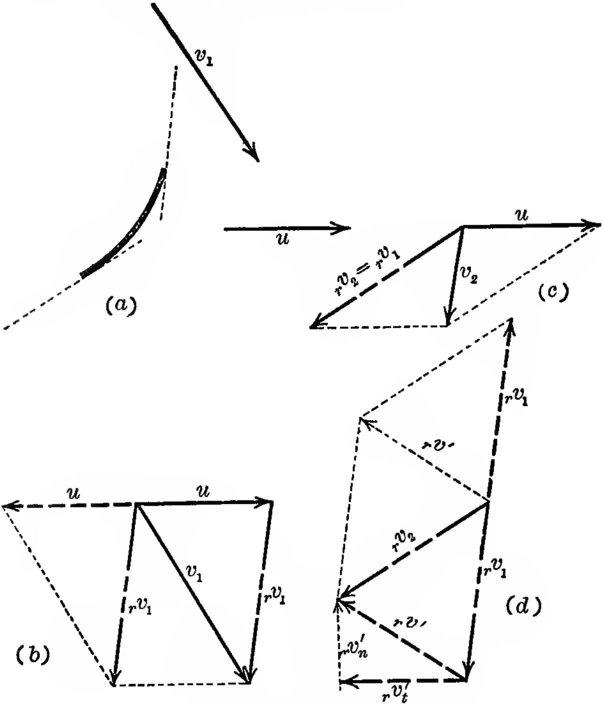

FIG. 87.

$_rv_t'$, the tangential component of this change in the relative velocity (in the direction of u);

$_rv_n'$, the normal component of $_rv'$ (in the direction perpendicular to u).

Let the directions and the magnitudes of v_1 and u be those shown in Fig. 87 (*a*) then the direction and magnitude of $_rv_1$ may be found by either method shown in Fig. 87 (*b*).

The water should always enter the vane without shock so as to avoid loss of energy and disturbance of flow through eddying or turbulent motion of the water. The tangent to the vane surface at entrance should therefore be parallel to $_rv_1$, as shown in Fig. 87 (*a*), so that the water may glide upon the vane surface with the least disturbance.

With a given shape of vane the direction of $_rv_2$ is fixed by the tangent to the vane at exit. Neglecting friction, the magnitude of $_rv_2$ equals the magnitude of $_rv_1$. The absolute velocity of exit can now be found as in Fig. 87 (*c*) by adding the motion of the vane to $_rv_2$.

The change in the relative velocity may be determined by either of the constructions shown in Fig. 87 (*d*).

EXERCISE 163. Draw the diagrams (*b*), (*c*), and (*d*) of Fig. 87, using the triangle construction for finding the difference between the velocities and superimposing the diagrams so as to avoid duplication of lines.

Show from this new diagram that **the change in the relative velocity equals the change in the absolute velocity.**

EXERCISE 164. Assume a given direction and magnitude for the absolute velocity of the jet and a given direction for the velocity of the vanes. Find graphically

(*a*) the velocity of the vanes,

(*b*) the change in the relative velocity of the water,

(*c*) the direction at entrance and at exit of the tangent to the vane surface,

if the water is to leave the vanes with no component of velocity in their direction of motion and with one-half of its initial absolute velocity.

EXERCISE 165. An axial-flow impulse wheel, fitted with a series of symmetrical vanes shaped as shown in Fig. 88, rotates about a vertical axis.

A jet delivering W pounds of water per second with a velocity v_1 at an angle α_1 enters the wheel on the upper side and leaves on the lower. Assume as known v_1, α_1, and u, the velocity of the vanes, neglect the energy due to the fall through the wheel and compute from the velocity diagrams,

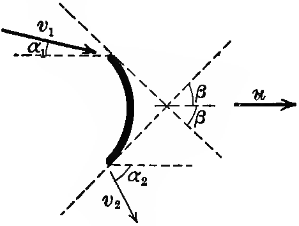

FIG. 88.

1. the relative velocity on entrance,
2. the slope of the blades at entrance (and exit), tan β,
3. the absolute exit velocity, v_2,
4. the slope of the exit velocity, tan α_2,
5. the power delivered to the vanes.

EXERCISE 166. (*a*) What value of u yields maximum power for given values of v_1 and α_1 in Exercise 165?

(*b*) When u has the value corresponding to maximum power, compute:

1. α_2,
2. tan β,
3. the efficiency,
4. the value of v_2.

EXERCISE 167. An axial-flow impulse wheel with vertical axis is supplied with water through a series of fixed guide vanes, Fig. 89. Assume $u = {}_r v_1 = {}_r v_2$. Find the angle β_1, in terms of α_1; also α_2, the angle at which the water leaves the wheel, in terms of β_2.

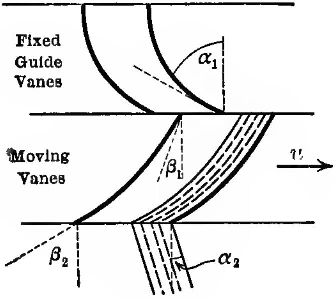

FIG. 89.

CHAPTER IX

RADIAL-FLOW IMPULSE WATER WHEELS

SECTION XXV

FUNDAMENTAL EQUATIONS

IN axial-flow impulse water wheels the radius at which the water acts upon the wheel is assumed to remain constant. In this chapter impulse wheels in which the water moves in such a manner as to increase or decrease the radius of its point of contact with the wheel will be discussed.

Fig. 90 represents diagrammatically a radial-flow impulse wheel. Here the water, led to the wheel through a vertical central pipe, is discharged through a fixed nozzle (or between fixed guide vanes) upon the moving vanes. The water then passes outward, increasing its radius of action, and leaves at the outer periphery of the wheel. The flow occurs in a plane perpendicular to the axis of rotation.

It should be noted that in impulse wheels the spaces between the moving vanes are not full of water and that the pressure in these spaces should be atmospheric. To insure this condition these wheels are "ventilated," i.e., air is admitted to the spaces between the moving vanes through holes (indicated at *a*, *a*, Fig. 90), cut through the crowns of the wheel.

The water while passing over the vanes of the wheel shown in Fig. 90 not only changes its velocity (in both direction and magnitude) but also changes its radius of action. It is therefore inconvenient to consider the forces exerted by the water from point to point along the vanes and then obtain the torque exerted by a summation of the moments of these forces. Instead the torque exerted by the water may be obtained directly by means of the fundamental equation

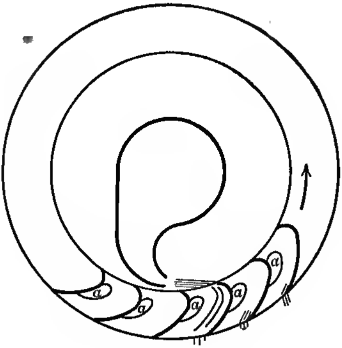

FIG. 90.

torque = moment of inertia × angular acceleration.

Let I = the moment of inertia of the water passing through the wheel per second;

$\Delta\omega$ = the change in angular velocity of this water;

and Δt = the time during which the water is in contact with the wheel;

then the torque exerted

$$T=(I\,\Delta\,t)\left(\frac{\Delta\,\omega}{\Delta\,t}\right)=I\,\Delta\,\omega.$$

Moment of inertia multiplied by the angular velocity is known as angular momentum. Therefore, the torque exerted on the wheel equals the rate of change of the angular momentum of the water passing through the wheel.

Change in Angular Momentum.—Change in angular momentum may be due to a change in angular velocity or to a change in moment of inertia or to a change of both ω and I.

A rigid system rotating about a fixed axis has a constant moment of inertia, therefore any change in angular momentum must be due to a change in angular velocity.

For instance, consider two masses $\left(\frac{m}{2}\right)$ concentrated at the ends of a massless rod, Fig. 91, rotating about a fixed, vertical axis, A, with an angular velocity ω.

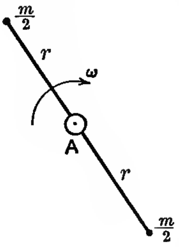

FIG. 91.

The moment of inertia of this system is

$$2\left(\frac{m}{2}\right)r^2 = m\,r^2, \text{ a constant.}$$

If the angular velocity of this system increases from ω_1 to ω_2 the change in angular momentum is

$$m\,r^2\,\omega_2 - m\,r^2\,\omega_1 = m\,r^2\,(\omega_2 - \omega_1).$$

If this change was produced in $\Delta\,t$ seconds, the average rate of change of angular momentum was

$$\frac{m\,r^2(\omega_2 - \omega_1)}{\Delta\,t},$$

and this equals the average external torque which must have acted upon the system during the time Δt.

As torque times angular displacement equals work done, the work per second or the power supplied to and stored in the system during the change must have been

$$\frac{m\,r^2(\omega_2-\omega_1)}{\Delta t}\left(\frac{\omega_2+\omega_1}{2}\right),$$

where $\frac{\omega_2+\omega_1}{2}$ is the average angular displacement per second during the change.

EXERCISE 168. Test all expressions in the above discussion by means of the theory of dimensions.

EXERCISE 169. Show that the work performed by the external torque during its action upon the above system equals the increase in kinetic energy due to the action of this torque.

A **deformable system** rotating about a fixed axis may have a variable moment of inertia. In such a system a change in angular momentum may be due to a change in the moment of inertia as well as to a change in the angular velocity of the system.

As an example consider the system shown in Fig. 92. Here the masses $\frac{m}{2}$ are attached to a spring, lying in the axis of rotation $A\,B$, and they are constrained to move radially relatively to the horizontal disc upon which they slide.

Assume the natural (unstressed) length of the spring such that the masses $\frac{m}{2}$ would lie at the center of the disc when the spring is unstressed. Then when the masses happen to be at a distance r from the center the force exerted by the spring upon them would be sr, where s is the strength of the spring.

When the system is rotating with an angular velocity

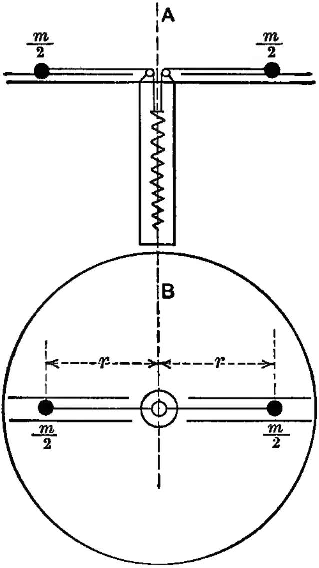

FIG. 92.

ω the centripetal force necessary to maintain each of the masses $\frac{m}{2}$ at a radius r is

$$\frac{m}{2}\omega^2 r,$$

and provided the spring alone furnishes this necessary force it would have to exert a tension of $m\,\omega^2\,r$ pounds when stretched r feet.

The strength of the required spring is thus the s in the equation

$$s\,r = m\,\omega^2\,r$$

or $$s = m\,\omega^2.$$

A spring of this strength will allow the masses $\frac{m}{2}$ to take any position provided the system revolves with a constant angular velocity

$$\omega = \sqrt{\frac{s}{m}}.$$

Exercise 170. If the masses $\frac{m}{2}$ weigh 50 pounds each and the system makes 300 r.p.m., what should be the strength of the spring?

The ideal, frictionless, massless $\left(\text{except for the masses } \frac{m}{2}\right)$ mechanism just described, having a spring adjusted to suit the masses and a certain angular velocity, would continue to rotate indefinitely with this constant angular velocity when once set in motion.

Assume now that a driving torque is applied to the system. This would naturally tend to increase the ω of the system. But with increasing ω a larger centripetal force must be applied to the masses $\frac{m}{2}$, and as this can only be obtained by an increased stretch of the spring the masses must move outward, seeking a new radius such that

$$s\,r = m\,\omega^2\,r,$$

where s and m are constant, and as $s = m\,\omega^2$, the angular velocity ω does not increase.

Thus we have the seemingly paradoxical condition of a system continuing to rotate with constant angular velocity even though an external torque acts upon it.

Similarly a resisting torque would not decrease the angular velocity of the system but would cause the masses $\frac{m}{2}$ to approach the center of the disc.

These conditions are readily explained when we note that any work performed by the driving torque is stored in the system as increased kinetic energy due to an increase in the moment of inertia of the system, and not to an increase in the angular velocity with constant moment of inertia, as is the case in a rigid system, as well as increased potential energy due to the stretching of the spring.

In this example the increase in kinetic energy due to a change of position of the masses $\frac{m}{2}$ from a radius r_1 to a radius r_2 is

$$(m\,r_2^2)\,\frac{\omega^2}{2} - (m\,r_1^2)\,\frac{\omega^2}{2} = \frac{m\,\omega^2}{2}(r_2^2 - r_1^2).$$

The increase in the potential energy of the spring is

$$\tfrac{1}{2}\,(s\,r_2 + s\,r_1)\,(r_2 - r_1) = \frac{s}{2}(r_2^2 - r_1^2) = \frac{m\,\omega^2}{2}(r_2^2 - r_1^2)$$

for $s = m\,\omega^2$.

Exercise 171. In the above example compute

(*a*) the change in angular momentum,

(*b*) the average torque exerted during the Δt seconds in which this change occurred,

(*c*) the work done by the torque.

As another example of a deformable system consider the following lecture experiment in physics. A man stands upon a turn-table holding a dumb-bell in each hand. The man with his arms outstretched laterally is given a certain angular velocity. When the man draws his hands inward his angular velocity at once increases **without** the action of an external torque. In this experiment both the moment of inertia and the angular velocity of the system change.

EXERCISE 172. A massless man upon a massless and frictionless turn-table changes the position of two masses (each equal to m) from a distance r_1 to a distance r_2 from the axis of rotation. The angular velocity of the system is initially ω_1 and $r_1 > r_2$.

(*a*) Compute the final angular velocity.

(*b*) Compute the increase in kinetic energy due to the change in the position of the masses in terms of ω_1, r_1, r_2, and m.

(*c*) Where does this increase in energy come from? Compute it at the source.

Thus it is evident that change in angular momentum should in general be expressed as

$$\Delta(I\,\omega)$$

and not in the special forms $I\,\Delta\omega$ or $\omega\,\Delta I$.

In radial-flow impulse wheels the moment of inertia of the particles of water changes with each change in radius and the angular velocity of these particles changes as well. Under these conditions **the torque exerted equals** the rate of change of the angular momentum of the water or, what amounts to the same thing, **the change in the angular momentum of the water supplied per second.**

SECTION XXVI

THE OUTWARD-FLOW IMPULSE WHEEL

Assume the outward-flow impulse wheel shown in Fig. 93 to lie in a horizontal plane. Let it be supplied with water from a head-race the free surface of which lies h_1 feet above the reference plane midway between the crowns

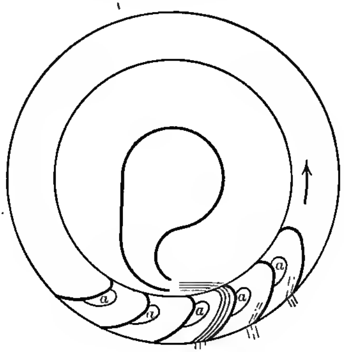

FIG. 93.

of the wheel. The flow through the moving vanes may be assumed to occur wholly in this reference plane.

If the transformation of the energy contained in one pound of water is followed from the free surface of the head-race to any point in the penstock leading to the nozzle we have

$$\frac{p_a}{w}+\frac{v_1^2}{2g}+h_1=\frac{p_2}{w}+\frac{v_2^2}{2g}+h_2+h_{l_2},$$

where h_{l_2} represents the sum of all the losses of energy due to entrance to and friction in the penstock up to the point designated as 2, p_2 represents the absolute pressure in the penstock at 2, and v_1 is the velocity of approach in the head-race. As this velocity of approach is small the term of the energy equation due to it is usually neglected.

The velocity with which the water leaves the nozzle and with which it enters the moving vanes is obtained by solving the equation

$$\frac{p_a}{w}+\frac{v_1^2}{2g}+h_1=\frac{p_a}{w}+\frac{v_3^2}{2g}+h_{l_3},$$

where the subscripts 3 designate the conditions at the end of the nozzle, for v_3.

We must now follow the water through the moving vanes.

Let ω be the angular velocity and

u_1 and u_2 be the peripheral velocities of the wheel (and vanes) at entrance and at exit respectively;

v_1 and v_2, the absolute velocities of the water at entrance and at exit;

${}_rv_1$ and ${}_rv_2$ the velocities of the water relative to the vanes at entrance and at exit;

α_1 and α_2 the inclinations of the absolute velocities to the radii;

β_1 and β_2 the inclinations of the relative velocities to the radii, all angles to be taken positive when lying on that side of the radius towards which the wheel is turning.

Then the angular velocity of the wheel

$$\omega=\frac{u_1}{r_1}=\frac{u_2}{r_2}.$$

As the water enters the wheel at a radius r_1 with an absolute velocity v_1 and as this point of the wheel moves with an absolute velocity, u_1, the relative velocity, ${}_rv_1$, of the entering water with respect to the wheel may be found by the construction shown in Fig. 94.

The direction of the arrow ${}_rv_1$ determines the tangent to the vane surface at entrance, for the water only glides on to the vanes without shock (and accompanying losses) if this condition is satisfied.

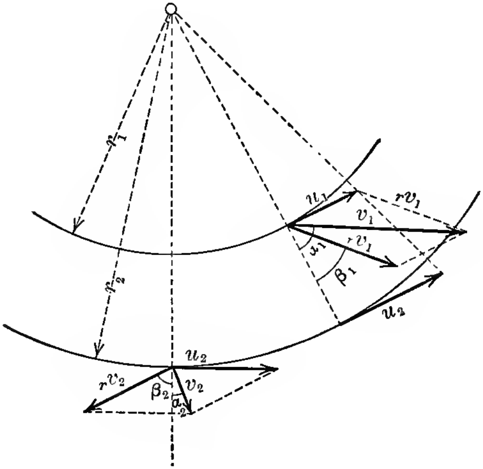

FIG. 94.

The direction of ${}_rv_2$ is determined by the tangent to the vane surface at exit. We have as yet no means of computing the magnitude of ${}_rv_2$. It is not equal to ${}_rv_1$ even if the frictional losses due to passage over the vanes is neglected. After the magnitude of ${}_rv_2$ has been determined the absolute velocity of exit v_2 may be found as shown in Fig. 94.

To compute ${}_r\mathbf{v}_2$ the law of conservation of energy is used, the kinetic energy lost by the water passing through the wheel per second is put equal to the work done by the moving water per second in driving the wheel.

Let W represent the weight of water passing through the wheel per second, then the kinetic energy lost by this water equals

$$\frac{W}{g}\frac{v_1^2}{2}-\frac{W}{g}\frac{v_2^2}{2}=\frac{W}{2g}(v_1^2-v_2^2).$$

In order to compute the work done by the water on the wheel the torque exerted by the water must first be computed. This torque equals the rate of change in the angular momentum of the water passing through the wheel with its sign changed.

The angular momentum of the water entering the wheel per second is

$$I\omega=\left(\frac{W}{g}r_1^2\right)\frac{v_1\sin\alpha_1}{r_1}=\frac{W}{g}r_1v_1\sin\alpha_1,$$

where the $\frac{v_1\sin\alpha_1}{r_1}$ is the angular velocity of the water and equals the tangential component of the velocity with which the water enters the wheel divided by the radius at which it enters.

Thus the rate of change of the angular momentum of the water passing through the wheel is

$$\frac{W}{g}r_2v_2\sin\alpha_2-\frac{W}{g}r_1v_1\sin\alpha_1.$$

This equals the torque exerted by the wheel on the water. The required torque exerted by the water on the wheel is evidently equal and opposite to the above torque and equals

$$\frac{W}{g}(r_1v_1 \sin \alpha_1 - r_2v_2 \sin \alpha_2).$$

This torque acting upon the wheel when multiplied by the angular velocity of the wheel (assumed constant) gives the work performed by the water per second which equals

$$\frac{W}{g}(r_1v_1 \sin \alpha_1 - r_2v_2 \sin \alpha_2)\omega = \frac{W}{g}(u_1v_1 \sin \alpha_1 - u_2v_2 \sin \alpha_2).$$

Neglecting the losses of energy due to shock and friction we may now put

$$\frac{W}{2g}(v_1^2 - v_2^2) = \frac{W}{g}(u_1v_1 \sin \alpha_1 - u_2v_2 \sin \alpha_2).$$

To introduce the relative velocities into this equation note from Fig. 94

that $$v_1^2 = u_1^2 + {}_rv_1^2 - 2u_1\,{}_rv_1 \cos(90 + \beta_1),$$

$$v_2^2 = u_2^2 + {}_rv_2^2 - 2u_2\,{}_rv_2 \cos(90 - \beta_2)$$

and that $$v_1 \sin \alpha_1 = u_1 + {}_rv_1 \sin \beta_1,$$

$$v_2 \sin \alpha_2 = u_2 - {}_rv_2 \sin \beta_2.$$

Substituting these values in the energy-work equation we have,

$$\tfrac{1}{2}(u_1^2 + {}_rv_1^2 + 2u_1\,{}_rv_1 \sin \beta_1 - u_2^2 - {}_rv_2^2 + 2u_2\,{}_rv_2 \sin \beta_2)$$

$$= u_1^2 + u_1\,{}_rv_1 \sin \beta_1 - u_2^2 + u_2\,{}_rv_2 \sin \beta_2,$$

or

$$_r\mathrm{v}_1^2 - {}_r\mathrm{v}_2^2 = \mathrm{u}_1^2 - \mathrm{u}_2^2.$$

This is the simplified form of the work-energy equation. By means of it ${}_rv_2$ may be computed.

EXERCISE 173. Develop the relation between the u's and the $_rv$'s for an inward-flow impulse wheel.

SECTION XXVII

EFFICIENCY AND DESIGN

The Efficiency of a hydraulic motor is defined as

$$\frac{\text{the energy utilized}}{\text{the available energy of the water}}.$$

Evidently the energy utilized may be considered to be either the work the motor is capable of doing or the work done by the water on the motor. In the first case the frictional losses of the motor are included in computing the efficiency and thus a **gross efficiency** is obtained. In the second case the efficiency is concerned only with the effectiveness of the wheel in withdrawing the energy from the water; this is called the **hydraulic efficiency.**

The available energy of the water may also be interpreted in several ways. Of course the total available energy due to a fall of h feet from the free surface of the head-race to the free surface of the tail-race is h foot-pounds per pound of water when the velocities of approach and of recession are neglected. If this available energy is used in computing the efficiency the result will be the efficiency of the whole system, i.e., penstock and motor. It may be, however, that only the efficiency of the wheel is in question; then the available energy should be the energy of the water leaving, or possibly the energy just before entering, the nozzles or guide vanes of the wheel.

Even when all frictional losses are neglected the efficiency of the wheel would not be unity, for the whole of the available energy is not removed from the water by the wheel. The water must clear the wheel and therefore must leave it with a certain velocity.

In the case of a frictionless impulse wheel running at such a speed that no energy is lost by shock the hydraulic efficiency of the wheel is

$$\eta=\frac{\frac{W}{g}\left(\frac{v_1^2}{2}-\frac{v_2^2}{2}\right)}{\frac{W}{g}\frac{v_1^2}{2}}=\frac{v_1^2-v_2^2}{v_1^2};$$

it also equals

$$\eta=\frac{(r_1v_1 \sin \alpha_1-r_2v_2 \sin \alpha_2)\omega}{\left(\frac{v_1^2}{2}\right)}.$$

EXERCISE 174. A radial-flow impulse wheel is so constructed that the water leaves the wheel radially when it runs at such a speed that no shock occurs at entrance; express its hydraulic efficiency, neglecting friction, in terms of

(*a*) u_1, v_1, and α_1;
(*b*) α_1, and β_1.

The Design of impulse wheels always involves the condition that the water must enter without shock.

In addition to this condition it is usual to assume either

1, that the water leaves the wheel radially; or

2, that the peripheral velocity of the wheel at exit equals the relative velocity of the water at exit.

Either of these conditions leads to a high efficiency, for under the first assumption the wheel extracts the whole tangential component of the motion of the water and under the second assumption the water leaves the wheel with a nearly radial velocity. These assumptions are made with the view of simplifying the necessary computations.

EXERCISE 175. In the design of an outward-flow impulse wheel the water is assumed to leave the wheel radially and v_1, r_1, r_2, α_1, β_2, and ω are assumed.

Draw the necessary diagrams and write the fundamental equation. From these show that the assumptions are incompatible.

Which one of the assumed quantities should be determined by the others?

EXERCISE 176. Assume r_1, r_2, v_1, α_1, (in practice 70° to 65°), β_2 (75° to 70°), and W for an outward-flow impulse wheel, also that $u_2 = {}_rv_2$.

(*a*) Draw the velocity diagrams.
(*b*) Find the angles of the vanes at entrance.
(*c*) Compute u_1 and the angular velocity of the wheel.
(*d*) Compute v_2.
(*e*) Compute the power delivered by the wheel.
(*f*) Compute the efficiency of the wheel.

EXERCISE 177. An outward-flow impulse wheel rated at 500 H.P. uses 8.5 cubic feet of water per second with an effective head of 570 feet. The diameters of the wheel are 7 feet 10.5 inches and 8 feet 11 inches. In this wheel $\alpha_1 = 57°$ and $\beta_2 = 75°$. The wheel is designed to run at 3.6 revolutions per second. What is the proper value for β_1? If the vanes have this angle at entrance what would be the ideal hydraulic efficiency of this wheel?

(Compute: v_1, u_1, β_1, ${}_rv_1$; u_2, ${}_rv_2$; v_2, and the efficiency.)

EXERCISE 178. (*a*) Does the wheel described in Exercise 177 satisfy either of the assumptions made on page 170?

(*b*) Compute the available horse-power of this wheel based upon the hydraulic efficiency computed in Exercise 177.

EXERCISE 179. Assume that the water enters an outward-flow impulse wheel radially, that $u_1 = \frac{v_1}{2}$, that $r_1 = .8\, r_2$, and that $\tan \beta_2 = 4$. Find v_2 in terms of v_1 by means of **graphical** constructions and determine the ideal hydraulic efficiency of the wheel.

EXERCISE 180. An outward-flow wheel radii r_1 and r_2, with plane, radial vanes revolves at such a speed that the water enters without shock. Compute the change in the kinetic energy of the water and the power delivered to the wheel in terms of v_1, α_1, r_1, r_2, and W. Note the sign of your result and explain its significance.

CHAPTER X

TURBINES

SECTION XXVIII

INTRODUCTION

IN an impulse wheel the pressure of the water is reduced to atmospheric pressure at the ends of the nozzles or guide vanes before the water reaches the moving vanes which extract its energy. Throughout the motion of the water over these moving vanes it is under atmospheric pressure. The nozzles thus convert the whole pressure-energy of the water reaching them into velocity-energy before the water enters the moving parts of the machine.

In a turbine the guide vanes (nozzles) convert only a part of the pressure-energy of the water passing through them into velocity-energy. Thus the water is still under a pressure greater than atmospheric pressure when it enters the moving vanes which extract its velocity-energy. Leakage must thus occur at the clearance between the fixed and the moving vanes. It is also evident that the passages between the moving vanes under these conditions must be completely filled with water and that water must be continuously admitted around the whole circumference to all passages between the moving vanes.

The pressure-energy remaining in the water on entrance to the moving vanes is not wasted. It is gradually converted into velocity-energy during its passage through these moving vanes and this velocity-energy is extracted by the moving vanes together with the velocity-energy originally present in the water reaching these vanes. The moving vanes thus serve both as nozzles and as extractors of kinetic energy.

Impulse wheels, as we have named them, are often called impulse turbines and in distinction turbines are called reaction turbines.

Before discussing turbines proper the earliest form of turbine, the simple

Reaction Turbine, as it is called, will be considered. This turbine has no fixed guide vanes. As shown in Fig. 95 the water enters a vertical pipe which also serves as a shaft about which the whole machine rotates. The water then passes into a low cylindrical vessel from which it escapes through two or more nozzles as shown at A or at B.

Assume that the nozzles are closed and that owing to some external means the machine rotates with a constant angular velocity ω. Then the pressure at the center of the cylindrical vessel is wh, but at any radius r it is not wh but has a greater value p. To compute this pressure p, write the equation for the radial equilibrium of an element of water rotating with the angular velocity of the machine, thus,

$$p\,b\,r\,d\theta + 2\left(p+\frac{dp}{2}\right)b\,dr\sin\frac{d\theta}{2} - (p+dp)\,b\,(r+dr)\,d\theta = \frac{w}{g}(b\,r\,d\,\theta\,dr)\omega^2 r,$$

where $\omega^2 r$ is the radial acceleration of the element of water considered.

This reduces to

$$r\,d\,p=\frac{w}{g}\,\omega^2\,r^2\,dr,$$

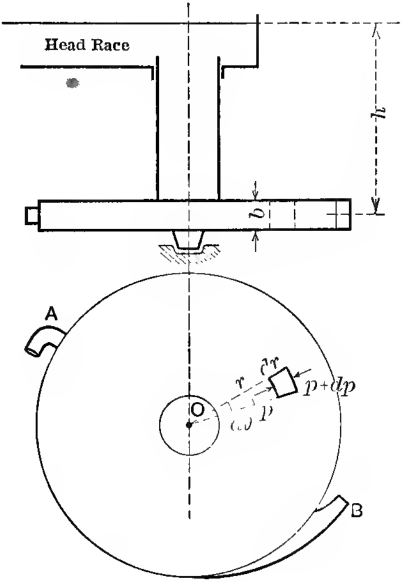

FIG. 95.

whence $$p=\frac{w}{g}\,\omega^2\frac{r^2}{2}+C.$$

But as $p=wh$ when $r=0$, $C=wh$,

and $$p=w\left(\frac{\omega^2\,r^2}{2g}+h\right)=w\left(\frac{u^2}{2g}+h\right),$$

where u is the peripheral velocity at the radius r at which the pressure is p.

A nozzle at a radius r_2 thus transforms a pressure

$$p_2=w\left(\frac{u_2^2}{2g}+h\right),$$

into velocity, and the relative velocity with which the water leaves the nozzle is

$$_rv_2=\sqrt{u_2^2+2\,gh.}$$

The absolute velocity with which the water leaves the wheel is thus

$$v_2={}_rv_2-u_2=\sqrt{u_2^2+2\,gh}-u_2.$$

A simpler method of computing the absolute velocity of discharge v_2 is based upon the law of conservation of energy.

If W represents the weight of water passing through the reaction turbine per second the potential energy supplied per second is Wh and the portion of this energy rejected per second as kinetic energy is $\frac{W}{g}\frac{v_2^2}{2}$. Thus neglecting frictional losses the energy extracted by the wheel per second is

$$Wh-\frac{W}{g}\frac{v_2^2}{2}.$$

The torque exerted on the water by the wheel equals the rate of change of the angular momentum of the water. The angular momentum of the water entering the machine per second is zero, on leaving it is

$$\left(\frac{W}{g}r_2^2\right)\left(\frac{v_2}{r_2}\right)=\frac{W}{g}r_2v_2.$$

The change is thus $\left(0-\frac{W}{g}r_2v_2\right)$ and this equals the torque exerted on the water by the turbine.

The torque exerted by the water on the turbine is

$$\frac{W}{g}r_2v_2,$$

and the energy absorbed by the wheel per second is

$$\left(\frac{W}{g}r_2v_2\right)\omega=\frac{W}{g}u_2v_2.$$

By the law of conservation of energy

we have, $$\frac{W}{g}u_2v_2=Wh-\frac{W}{g}\frac{v_2^2}{2},$$

from which $$v_2=\sqrt{u_2^2+2\,gh}-u_2.$$

Exercise 181. Show that the ideal hydraulic efficiency of the reaction turbine is

$$\frac{u_2(\sqrt{u_2^2+2\,gh}-u_2)}{gh}.$$

At what speed must a reaction turbine be run in order to yield maximum efficiency?

In a reaction turbine the actual relative velocity of discharge through the nozzle is not $\sqrt{u_2^2+2\,gh}$. Making allowance for the frictional losses by means of a coefficient of velocity c_v we have

$$_rv_2=c_v\sqrt{u_2^2+2\,gh},$$

and the efficiency becomes

$$\frac{(c_v\sqrt{u_2^2+2\,gh}-u_2)u_2}{gh}.$$

EXERCISE 182. Show that this efficiency becomes a maximum when

$$u_2^2 = gh\left\{\frac{1}{\sqrt{1-c_v^2}} - 1\right\}$$

and that the maximum efficiency is

$$1-\sqrt{1-c_v^2},$$

and that this reduces to 0.66 when $c_v = 0.94$.

SECTION XXIX

FUNDAMENTAL EQUATIONS

To fix our ideas consider as a typical example of a turbine the outward-flow turbine shown in Fig. 96.

Here the water is led by the penstock from the head-race to the stationary annular casing containing the fixed guide vanes. The spaces between these vanes act as nozzles (transforming pressure-energy into velocity-energy) and deliver the water, at the proper angle, to the moving vanes.

After passing through the moving vanes, during which a further transformation of pressure into velocity takes place and during which the kinetic energy of the water is extracted by these moving vanes, the water is discharged either into the air above the tail-race as shown in Fig. 96 or into a draft tube which in turn discharges below the free surface of the tail-race.

The relations between the absolute and the relative velocities at entrance to and at exit from the moving vanes are shown in Fig. 97. The notation is the same as that used for impulse wheels on page 165.

As all passages tnrough a turbine are always full of water the **equation of continuity** applies to the flow through the moving vanes as well as to the flow through the penstock and through the guide vanes. This is the first fundamental equation.

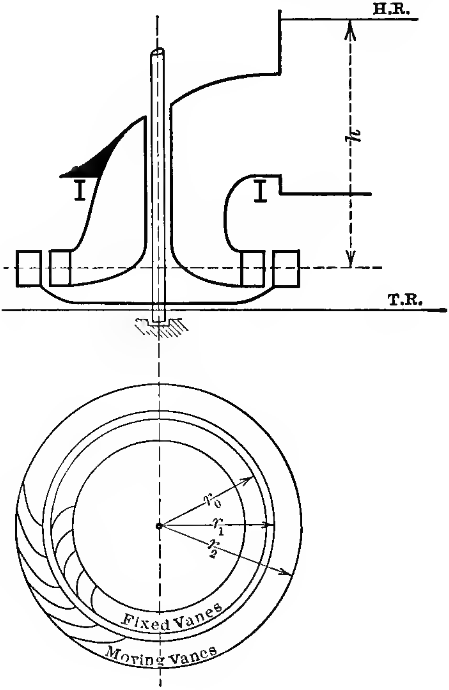

Fig. 96.

Let A_1 represent that part of the cylindrical surface whose radius is r_1, through which the water enters the moving part of the turbine, and

A_2, the corresponding area at exit;

Q, the quantity of water passing through the turbine per second; then the equation of continuity for the passages through the moving blades is

$$Q = A_1\,{}_r v_1 \cos\beta_1 = A_2\,{}_r v_2 \cos\beta_2 = A_1 v_1 \cos\alpha_1 = A_2 v_2 \cos\alpha_2.$$

In these equations the areas A_1 and A_2 are multiplied by the component velocities normal to the surfaces considered. If the actual velocities relative to the moving part of the turbine are used then the areas normal to the direction of flow would be the multipliers. Fig. 98 shows that this method of computing the flow gives the same results as above obtained.

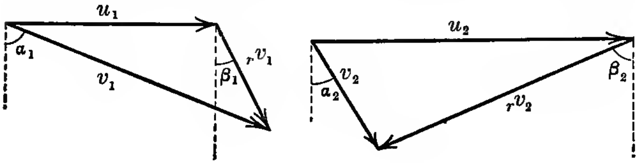

FIG. 97.

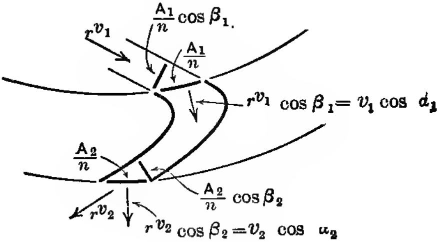

FIG. 98.

The second fundamental equation is the **equation of energy.**

Let point o be any point in the channels leading to the moving vanes and point 3 be any point in the channels leading the water from the moving vanes to the tail-race. To fix our ideas point o may be any point in the penstock (Fig. 96) and in this figure point 3 would correspond to the point at which the water leaves the wheel, as in this illustration no draft tube is used to lead the water from the rotor to the tail-race. The theory of the draft tube will be explained on page 183.

The energy in the water at point o is

$$\frac{p_0}{w}+\frac{v_0^2}{2g}+h_0,$$

and at point 3

$$\frac{p_3}{w}+\frac{v_3^2}{2g}+h_3.$$

These energies are evidently not equal, but their difference must equal the energy wasted in overcoming the resistances between points o and 3 plus the energy extracted by the rotor of the turbine from the water.

Let, h_f represent the head or the energy per pound wasted in overcoming resistance between 1 and 3;

and h_w the head or the energy per pound extracted by the rotor.

Then
$$\left(\frac{p_0}{w}+\frac{v_0^2}{2g}+h_0\right)-\left(\frac{p_3}{w}+\frac{v_3^2}{2g}+h_3\right)=h_f+h_w.$$

Now, h_w may be computed from the torque exerted by the water on the wheel and this torque equals the negative of the change in angular momentum of the water passing through the wheel per second.

Thus the torque exerted on the wheel by the water equals

$$-\left\{\frac{W}{g}r_2^2\frac{v_2 \sin \alpha_2}{r_2}-\frac{W}{g}r_1^2\frac{v_1 \sin \alpha_1}{r_1}\right\}$$

$$=\frac{W}{g}\{r_1v_1 \sin \alpha_1 - r_2v_2 \sin \alpha_2\},$$

where W equals the pounds of water passing through the wheel per second.

The work performed by the water per second equals

$$\frac{W}{g}(r_1v_1 \sin \alpha_1 - r_2v_2 \sin \alpha_2)\omega = \frac{W}{g}(u_1v_1 \sin \alpha_1 - u_2v_2 \sin \alpha_2),$$

and the energy extracted by the rotor per pound of water equals

$$\frac{1}{g}\{u_1v_1 \sin \alpha_1 - u_2v_2 \sin \alpha_2\} = h_w.$$

Neglecting frictional losses the energy equation becomes

$$\left(\frac{p_0}{w}+\frac{v_0^2}{2g}+h_0\right)-\left(\frac{p_3}{w}+\frac{v_3^2}{2g}+h_3\right)=\frac{1}{g}(u_1v_1 \sin \alpha_1 - u_2v_2 \sin \alpha_2).$$

If we now consider the flow through the rotor of the turbine the point 0 would become the point 1, the point 3 would become the point 2, and the energy equation becomes

$$\left(\frac{p_1}{w}+\frac{v_1^2}{2g}+h_1\right)-\left(\frac{p_2}{w}+\frac{v_2^2}{2g}+h_2\right)=\frac{1}{g}(u_1v_1 \sin \alpha_1 - u_2v_2 \sin \alpha_2).$$

In Fig. 96, $h_1=h_2$, $p_2=p_a$, the atmospheric pressure; p_1 is the absolute pressure in the clearance space between the fixed and the moving vanes.

EXERCISE 183. Show that the energy equation becomes

$$h-\frac{v_2{}^2}{2g}=\frac{1}{g}(u_1v_1\sin\alpha_1-u_1v_1\sin\alpha_2)$$

when it conforms with the conditions in Fig. 96.

The energy equation as applied to the flow through the rotor of the turbine may also be expressed in terms of the relative velocities of the water by means of the relations between the absolute and the relative velocities shown in Fig. 97 (see page 168).

EXERCISE 184. Make this transformation.

This transformation gives

$$\left(\frac{p_1}{w}+\frac{{}_rv_1{}^2-u_1{}^2}{2g}+h_1\right)-\left(\frac{p_2}{w}+\frac{{}_rv_2{}^2-u_2{}^2}{2g}+h_2\right)=0.$$

Note that in turbines p_1 is not equal to p_2 as in the case of outward-flow impulse wheels. Thus the equation ${}_rv_1{}^2-{}_rv_2{}^2=u_1{}^2-u_2{}^2$, developed for radial-flow impulse wheels, does not hold for turbines.

EXERCISE 185. To what form does the energy equation, in terms of the relative velocities as above deduced, reduce for (*a*) radial-flow impulse wheels, (*b*) axial-flow impulse wheels?

The Draft Tube.—When a turbine discharges above the level of the free surface of the tail-race as shown in Fig. 96 the only part of the total fall (from the level of the head-race to the level of the tail-race) that the turbine can possibly utilize is the head h from the level of the head-race to the horizontal plane midway between the crowns of the wheel.

To avoid this loss of available head the turbine may be submerged so that it discharges under the level of the tail race. This construction renders the turbine inaccessible for inspection and adjustment.

Instead of submerging the turbine its rotor may be enclosed in a fixed casing which forms part of a tube leading from the turbine to below the level of the tail-race. This tube is called the draft or the suction tube. This tube may be made annular in section and it must contain no sudden changes in section.

Let the subscript 2 apply to the conditions at the exit of the moving vanes and the subscript 3 to the conditions in the suction tube at a level with the free surface in the tail-race,

then
$$\frac{p_2}{w}+\frac{v_2^2}{2g}+h_2=\frac{p_3}{w}+\frac{v_3^2}{2g}+h_3,$$

were $p_3=p_a$, the atmospheric pressure, $h_3=0$ if the datum plane is the level in the tail-race, and h_2 is the height of the outlets of the moving vanes above the level in the tail-race.

Thus
$$\frac{p_2}{w}+\frac{v_2^2}{2g}+h_2=\frac{p_a}{w}+\frac{v_3^2}{2g}.$$

If the exit section of the suction tube equals the combined exit areas of the moving vanes measured normally to v_2 then $v_3=v_2$ by reason of the equation of continuity; the absolute pressure head against which the water is discharged from the moving vanes becomes

$$\frac{p_2}{w}=\frac{p_a}{w}-h_2.$$

As $\frac{p_2}{w}$ cannot be less than zero, h_2 cannot be greater than

$\frac{p_a}{w} = 34$ feet. Experience shows that h_2 cannot exceed 25 feet under actual conditions.

Under ideal conditions a turbine discharging into a draft tube 34 feet long discharges into a perfect vacuum. If the total fall is h feet the available head at the turbine is only $h - 34$ feet, but the back pressure is zero. If the turbine be placed at the level of the tail-race the available head would be h feet, but the back pressure would be p_a, the atmospheric pressure. The draft tube thus does not increase the total available head, but simply changes the distribution of the pressures in the system and permits the utilization of a greater part of the total head.

When $A_2 \cos \alpha_2 = A_3$, $v_2 = v_3$ and the kinetic energy carried away by the discharged water per second is the same in both of the conditions assumed in the preceding paragraph. In a well-designed turbine v_2 is made as small as possible consistent with necessary flow to yield the power required at the least cost of installation. The efficiency of an existing turbine may, however, be improved by reducing the final velocity of discharge from the suction tube v_3 by gradually enlarging the section of the draft tube as the level of the tail water is approached. Thus A_3 becomes larger than $A_2 \cos \alpha_2$ and v_3 less than v_2. Under these conditions the turbine cannot be placed higher than

$$h_2 = \frac{p_a}{w} - \frac{v_2^2}{2g} + \frac{v_3^2}{2g} \text{ feet}$$

above the tail water.

Exercise 186. Prove the last statement and show that the velocity of the water leaving a turbine supplied with a draft tube which increases in sectional area towards the tail-race is greater than the exit velocity of the water from the same turbine when attached to a draft tube of constant diameter, the turbine in both cases discharging against zero back pressure.

The velocity of the water entering a turbine depends not only upon the available head at the outlet of the passages between the guide vanes, but also upon the dimensions of the turbine and upon the speed at which it runs. The difference between impulse wheels and turbines in this respect should be carefully noted. In the impulse wheel the pressure of the water reduces to the atmospheric pressure on leaving the nozzles and the absolute velocity with which the water reaches the wheel is wholly independent of the speed of the wheel. Also the passages in the wheel being never full of water their dimensions cannot affect the initial velocity of the water entering the rotor.

Let us compute the absolute velocity of the water leaving the guide vanes and entering the rotor of a turbine. All calculations on turbines must be based upon the energy equation, the equation of continuity, and the triangles of velocities, Fig. 97.

Neglecting friction and referring to Fig. 96 the energy equation is obtained by placing the energy lost per pound of water equal to the work extracted from each pound of water by the rotor,

thus $$h-\frac{v_2^2}{2g}=\frac{1}{g}(u_1v_1 \sin \alpha_1-u_2v_2 \sin \alpha_2).$$

The design of the wheel fixes the angle α_1 but not α_2. Thus to find v_1, not only v_2 but also α_2 must be eliminated.

Fig. 97 shows that

$$v_2 \sin \alpha_2=u_2-{}_rv_2 \sin \beta_2$$

and that $v_2^2 = u_2^2 + {}_rv_2^2 - 2u_2\, {}_rv_2 \sin \beta_2,$

so the energy equation becomes

$$2\,gh = (u_2^2 + {}_rv_2^2 - 2u_2\, {}_rv_2 \sin \beta_2) + 2\{u_1v_1 \sin \alpha_1 - u_2(u_2 - {}_rv_2 \sin \beta_2)\}$$
$$= {}_rv_2^2 + 2\, u_1v_1 \sin \alpha_1 - u_2^2.$$

The equation of continuity shows that

$$A_2\, {}_rv_2 \cos \beta_2 = A_1v_1 \cos \alpha_1,$$

and of course $\frac{u_1}{r_1} = \frac{u_2}{r_2},$

so that

$$2\,gh = \left(\frac{A_1 \cos \alpha_1}{A_2 \cos \beta_2}\right)^2 v_1^2 + 2u_1v_1 \sin \alpha_1 - \left(\frac{r_2}{r_1}\right)^2 u_1^2,$$

where β_2 is determined by the design of the wheel. Thus v_1 depends not only on h, but upon A_1, A_2, r_1, r_2, α_1, β_2 and upon the speed of the wheel, u_1.

SECTION XXX

DESIGN OF TURBINES

The design of a turbine should of course be such as to reduce all losses as much as possible and thus obtain a high efficiency.

One of the important losses is that due to impact or shock with the attending disturbance of smooth flow when the direction of the relative velocity of the water entering the rotor does not coincide with the tangent to the vane surfaces at entrance. As a change in the speed of the rotor changes not only ${}_rv_1$ in direction and magnitude, but also v_1, it is evident that complete avoidance of abrupt change of velocity at entrance can only be attained for a given fixed speed for any one turbine.

Another unavoidable loss is due to the absolute velocity v_2 with which the water leaves the rotor. This velocity should always be small.

In order to simplify the energy equation and at the same time produce conditions leading to high efficiency it is usual to assume either one of the two following conditions in designing turbines, namely:

Either, 1, that the water leaves the rotor radially so that $\alpha_2 = 0$, or

2, that the relative velocity of the water at exit equals the peripheral velocity of the wheel at its exit circumference so that $u_2 = {}_rv_2$.

In either case the angle which the relative velocity of exit makes with the radius must be large.

The design of turbines involves many assumptions other than the above. These rest upon the results of experience and experimentation. Thus the ratio of A_1 to A_2 is usually assumed to lie between

0.5 and 1.0 for outward-flow turbines,
0.6 and 1.5 for inward-flow turbines,

and is of course equal to unity for axial-flow turbines. Also the ratio of r_1 to r_2 is assumed to lie between

0.7 and 0.85 for outward-flow,
1.2 and 2.0 for inward-flow,

and equals unity for axial-flow turbines.

The angle α_1 is usually assumed to lie between

65° and 75° for outward-flow,
and 65° and 80° for inward-flow turbines.

In computations it is convenient to express the absolute velocity v_1 with which the water enters the rotor as a fraction, K, of the velocity due to the available head h and express all quantities in terms of K.

Thus $$v_1=K\sqrt{2\,gh}.$$

In impulse wheels K equals unity.

Exercise 187. An outward-flow turbine is to be designed so that the water enters the rotor without shock and leaves it radially (with no peripheral component of velocity). The moving vanes are assumed to have a radial direction at their inner ends and h, α_1, r_1, r_2, A_1, A_2 are assumed to be fixed by other considerations.

(*a*) Draw the velocity triangles for entrance to and for exit from the rotor.

(*b*) Write the energy equation.

(*c*) Write the equation of continuity.

(*d*) Compute the absolute velocity with which the water enters the rotor; thus finding K.

(*e*) Compute the peripheral velocity of the rotor at a radius r_1 in terms of K.

(*f*) Compute the absolute velocity of exit of the water.

(*g*) Compute the ideal hydraulic efficiency of the turbine.

(*h*) Compute the proper angle of the vanes at exit.

(*i*) Compute the quantity of water passing through the turbine per second.

(*j*) Compute the horse-power developed by the turbine, friction neglected.

Exercise 188. In an outward-flow turbine the water enters the rotor without shock and leaves it radially. Assuming α_1, β_2, h, r_1, r_2, A_1, A_2 as known. Compute

(*a*) the value of K in $v_1=K\sqrt{2\,gh}$,

(*b*) the speed of the rotor,

(*c*) the efficiency, neglecting friction,

(*d*) the angle of the moving vanes at entrance.

Exercise 189. An outward-flow turbine is designed so that the water enters the rotor without shock and leaves it radially when the rotor makes 250 revolutions per minute. The turbine is to operate under an effective head of 140 feet and utilize 200 cubic feet of water per second. The diameters of the rotor are 5 and 6 feet and it contains 32 vanes 1 inch thick at inlet and 1.25 inches thick at outlet both measured along the circumference. The depth between the crowns is 10 inches.

(*a*) Compute the ideal hydraulic efficiency and the horsepower, neglecting frictional losses.

(*b*) At what angle should the water leave the guide vanes?

(*c*) What should be the entrance and the exit angles of the moving vanes?

Exercise 190. Compute the pressure head between the fixed and the moving vanes in the turbine described in Exercise 189.

Exercise 191. The clearance between the casing and the rotor in the turbine described in Exercise 189 is 1/16 inch; compute the leakage through this clearance, assuming 0.60 as the coefficient of discharge.

Exercise 192. Each vane in the rotor of the turbine described in Exercise 189 subtends a central angle of 15°. Assume the average radial velocity of the water through the rotor to be the mean between the entrance and the exit radial velocities and draw the absolute path of the water during its passage through the rotor for an assumed vane form which satisfies the results obtained in Exercise 189.

Instead of assuming that the water leaves the turbine radially (so that $\alpha_2=0$) the assumption that the peripheral speed of the rotor at exit equals the relative velocity of the leaving water ($u_2={}_rv_2$) is often made. The water under the latter assumption leaves the rotor in a nearly radial direction and with small absolute velocity, when the angle β_2 is large. The velocity triangle at exit (Fig. 99) illustrates these statements. From this figure we find analytically that,

$$v_2 = 2\,u_2 \sin\gamma_2, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

where

$$\gamma_2 = 45 - \frac{\beta_2}{2}.$$

The energy equation when friction losses are neglected is

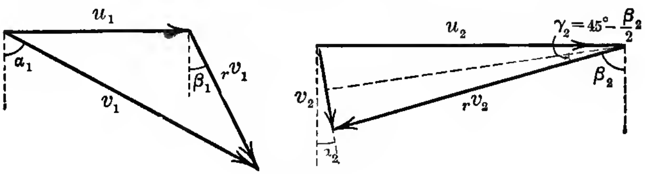

FIG. 99.

$$\frac{1}{g}(u_1v_1 \sin\alpha_1 - u_2v_2 \sin\alpha_2) = h - \frac{v_2^2}{2g}.$$

Under the assumed conditions it becomes

$$\frac{1}{g}(u_1v_1 \sin\alpha_1 - 2\,u_2^2 \sin\alpha_2 \sin\gamma_2) = h - \frac{4u_2^2 \sin^2\gamma_2}{2g},$$

and if we note from Fig. 99

that

$$\gamma_2 = \alpha_2,$$

the energy equation reduces to the following simple form

$$u_1v_1 \sin\alpha_1 = gh. \quad . \; . \; . \; . \; . \; . \quad (2)$$

The equation of continuity is

$$A_1v_1 \cos\alpha_1 = A_2v_2 \cos\alpha_2 = A_2\,{}_rv_2 \cos\beta_2$$

but as u_2 now equals $_rv_2$ we may also place

$$A_1v_1 \cos\alpha_1 = A_2u_2 \cos\beta_2. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

If we assume that h, r_1, r_2, A_1, A_2, α_1, and β_2 have been fixed by other considerations there remain to be determined the velocity with which the water leaves the guide vanes, the speed at which the turbine should run, and the angle of the vanes at entrance, or v_1, u_1, and β_1.

The equations available are always

1. The energy equation;
2. The equation of continuity, and
3. The conditions made necessary by the triangle of velocities at entrance so as to avoid shock when the water passes from the fixed to the moving vanes.

In the case under consideration these equations are

$$u_1 v_1 \sin \alpha_1 = gh, \quad \dots \dots \dots \quad (4)$$

$$A_1 v_1 \cos\alpha_1 = A_2 u_2 \cos \beta_2, \quad \dots \dots \quad (5)$$

and

$$u_1 \cos \beta_1 = v_1 \sin (\alpha_1 - \beta_1), \quad \dots \dots \quad (6)$$

the last obtained by the law of sines from Fig. 99.

In these equations the only unknown quantities are u_1, u_2, v_1, and β_1 and as u_1 and u_2 are connected by the equation

$$\frac{u_1}{r_1} = \frac{u_2}{r_2}, \quad \dots \dots \dots \quad (7)$$

all unknowns can now be determined.

Thus from Equation (5) with the aid of Equation (7)

$$u_1 = \frac{r_1}{r_2} \frac{A_1}{A_2} \frac{\cos \alpha_1}{\cos \beta_2} v_1.$$

Substituting in Equation (4)

we find,

$$v_1 = \sqrt{\frac{r_2}{r_1} \frac{A_2}{A_1} \frac{\cos \beta_2}{\sin 2\,\alpha_1} (2\,gh)}$$

$$= K\sqrt{2\,gh},$$

where $$K=\sqrt{\frac{r_2}{r_1}\frac{A_2}{A_1}\frac{\cos\beta_2}{\sin 2\alpha_1}}.$$

From Equation (4)

$$u_1=\frac{gh}{v_1\sin\alpha_1},$$

thus the speed of the turbine is determined.

Instead of solving Equation (6) for β_1 return to Fig. 99 and note that

$$\tan\beta_1=\frac{v_1\sin\alpha_1-u_1}{v_1\cos\alpha_1}.$$

Exercise 193. Compute the discharge, the absolute velocity with which the water leaves the turbine, and the efficiency of a turbine when $u_2={}_rv_2$, in terms of the quantities assumed in the above example.

PROBLEMS FOR REVIEW

194. The center of a circular disc 6 feet in diameter is submerged to a depth of 10 feet in sea-water ($w=65$). How far from the center does the resultant force of the water act on one face of the disc (*a*) when the disc lies in a vertical plane, (*b*) when the disc lies in a horizontal plane?

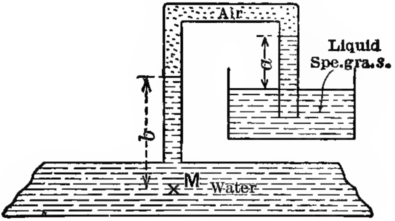

FIG. 100.

195. Find the gage pressure at M (Fig. 100) in terms of w, b, a, and s.

196. The pressure gage on a hot-water heating system reads 30 pounds per square inch when the system is cold.

(*a*) How high above the gage does the water extend?

(*b*) Will the gage indicate a higher or lower pressure when the system is hot? Give the reasons for your answer.

197. Show that the pressure p_1 must equal both p_2 and p_3, if the prismatic solid is to be in equilibrium under the action of these pressures, Fig. 101.

198. The top of the tank, Fig. 102, contains air at 30 pounds per square inch absolute. If the manometer is filled with mercury, specific gravity 13.6, compute x when the barometer reads 29 inches.

199. (*a*) Deduce a formula for the location of the center of pressure of a plane surface in terms of the first and the second moments of area of the surface whose center of pressure is sought.

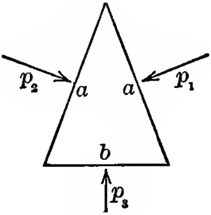

FIG. 101.

(*b*) About what axes should these moments be computed?

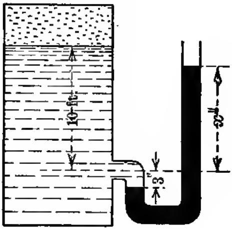

FIG. 102.

200. An open cylindrical tank 6 feet in diameter placed upon a flat-car is filled with oil (specific gravity 0.9) to within 2 feet of its brim. If the car moves on a straight track with an acceleration of 3 feet per second, does the oil spill?

201. How deep must the top of a rectangular plate 2 feet by 1 foot be sunk with its longer edge vertical so that the distance

between its center of area and its center of pressure may be 3 inches?

202. Give a simple formula for obtaining the force exerted upon any plane surface by the weight of a liquid and prove your formula.

203. A tank in the form of a right circular cylinder $2r$ feet in diameter lies with its axis horizontal. The tank is filled with a liquid of specific weight w.

(*a*) Find the force exerted by the liquid upon one end of the tank.

(*b*) Locate the point of application of this force.

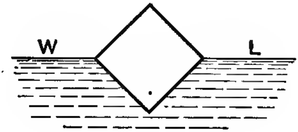

FIG. 103.

204. The pressure in a vertical gas main at a certain point is 2 inches of water above the atmospheric pressure at that level. What is the pressure in the main at a point 100 feet higher up, measured in inches of water?

Assume the specific weight of air to be constant and equal to .08 pound per cubic foot; also the specific weight of gas to be constant and equal to .05 pound per cubic foot.

205. Compute the force exerted by water upon the hemispherical surface of a hemisphere immersed with its base horizontal, and upward, and 10 feet below the surface. The radius of the spherical surface is 5 feet.

206. A caisson (in the form of a box without a cover), 40 feet long, 20 feet wide, and 25 feet deep, weighing 140,000 pounds, is to be sunk in water 22 feet deep.

(*a*) How deep will this caisson sink when unloaded?

(*b*) What load will be required to sink it to the bottom?

207. Is the equilibrium of a homogeneous rectangular parallelepiped $b \times b \times l$, specific gravity 0.5, stable, unstable, or neutral when it floats in water in the position shown in Fig. 103? Give the reasons for your answer.

208. Develop the formula for the distance between the center of buoyancy and the metacenter of any body which is symmetrical about a vertical plane.

209. A right cone (radius of base r, altitude h) is immersed in water with its base in the free surface. Compute the force exerted upon the cone by the water.

210. How deep must a square plate a feet square be submerged in a vertical plane so that the force exerted by the liquid acts at $\frac{a}{12}$ feet below the center of area?

211. "The total pressure on a curved surface." State clearly your conception of the meaning of this phrase.

212. Find the equation of the free surface of a liquid rotating about a vertical axis with an angular velocity ω.

213. (*a*) Show by means of a carefully drawn sketch the nature of a jet issuing through a standard orifice.

(*b*) In what portion of the jet is the pressure greater than atmospheric?

(*c*) Why is the pressure in this portion greater than atmospheric?

214. Develop an expression for the discharge through an orifice. State clearly the meaning of each letter used.

215. State: (*a*) Torricelli's theorem;
(*b*) Archimedes' principle.

216. Describe a standard orifice and explain why the greatest velocity of efflux occurs at the vena contracta.

217. A horizontal pipe 6 inches in diameter terminates in a nozzle which reduces the diameter to 2 inches. The pressure at the base of the nozzle during discharge is 60 pounds per square

inch, gage ($c_c = 1$; $c_v = 0.98$). Compute the velocity of the jet leaving the nozzle.

218. Explain (with sketches) the terms coefficient of velocity, coefficient of contraction, and coefficient of discharge as applied to the discharge through an orifice.

219. A standard orifice $\frac{1}{2}$ inch in diameter discharges .03 of a cubic foot of water per second. Compute the head on the center of the orifice.

220. Between what limits must the specific gravity of a right circular cone lie so that it may float in water with its vertex submerged and its base horizontal?

221. Compute the resultant force exerted by the water upon the cone described in 220 when it floats in neutral equilibrium.

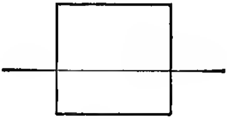

FIG. 104.

222. What must be the specific gravity of a prism of square cross-section in order that it may float in neutral equilibrium in the position shown in Fig. 104?

223. A jet of water under a head of 6 feet enters a condenser in which the absolute pressure is 4 pounds per square inch. The atmospheric pressure is 14.7 pounds per sq.in. Find the velocity of the jet, neglecting friction and assuming zero velocity at the surface of the supply tank.

224. A homogeneous cylinder is fastened to the end of a tank as shown in Fig. 105. The cylinder is free to revolve about the fixed axis *AB* and the tank is filled, to the level *CD*, with water. The inventor of this machine quotes the law of Archimedes and states that, as the portion of the cylinder immersed in the water is lighter than the rest of the cylinder, the cylinder

should rotate about the axis AB in the direction shown by the arrow.

Explain with the aid of a sketch the truth or the falsity of this contention. Assume the machine frictionless and that no leakage occurs between the cylinder and tank.

225. A beam 6″×6″×10′, specific gravity 0.60, floats in water. Find the depth of flotation.

226. Define hydraulics.

227. Prove Bernoulli's theorem.

228. Describe briefly the Venturi water meter and explain the principle on which its action depends.

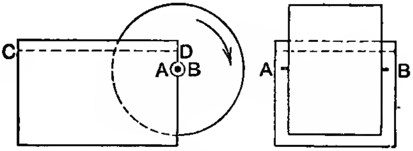

FIG. 105.

229. Show that the pressure in a mouthpiece may be less than the atmospheric pressure against which it discharges.

230. Show that each term in the formula expressing Bernoulli's theorem represents energy.

231. Prove Torricelli's theorem.

232. A sphere is placed in a cup, Fig. 106, to the bottom of which water is supplied under pressure. Describe and account for the resulting phenomenon.

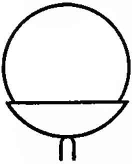

FIG. 106.

233. What time is required to empty a cylindrical vessel (horizontal section constant and equal to A), containing h feet of water, through an orifice (section a, coefficient of discharge c) in its bottom?

234. A tank contains air and water under a pressure of 100 pounds per square inch gage. Compute the initial velocity of

efflux through a standard orifice whose center is 10 feet below the level of the free surface of the water in the tank.

235. A Venturi meter has a diameter of 4 feet at the pipe and 1.25 feet in the throat. The pressure head at the pipe is 100 feet and at the throat 90 feet when water is flowing through the meter. Compute the flow through the meter.

236. A tank 600 square feet in area discharges through an orifice 1 square foot in area. How long will it take for the level to fall from 50 feet to 25 feet above the orifice? (Coefficient of discharge = 0.60.)

237. A flaring pipe is to be attached to an orifice (area 1 square inch) upon which the head is 20 feet. How large may the area of discharge be made, the flow to remain steady?

238. Compute the time required to discharge the water filling the tank (Fig. 107) through an orifice whose area is *a* square feet.

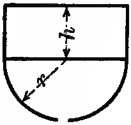

FIG. 107.

239. Some of the water contained in a cylindrical tank was discharged through an orifice. The head on the orifice fell from 12 feet to 3 feet during the discharge. What constant head on the orifice would cause the same volume of water to flow in the same time?

240. Compute the flow of water through the pipe in Fig. 108 (sp.gr. of Hg 13.6).

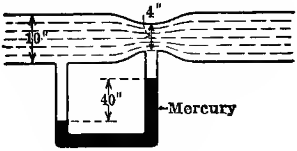

FIG. 108.

241. Does Fig. 108 contain sufficient data to allow the computation of the pressures in the pipe? Give the reasons for

your answer and if possible compute the pressure at the throat. Neglect the effect of the water filling the branches of the U-tube.

242. A pipe 6 inches in diameter terminates in a cap containing a standard orifice 2 inches in diameter. The pressure of the water in the pipe 10 feet above the orifice is 40 pounds per square inch gage. Compute the velocity of flow in the pipe and the discharge in gallons per second. ($c=0.60$.)

243. A standard orifice 1 square inch in area discharges under a constant head of 100 feet. Compute the velocity of approach in a pipe one foot in diameter leading to the orifice.

244. The throat of a Venturi meter has $\frac{1}{3}$ the area of the main. The area of the main is A square feet. Find the difference in the pressures in the main and in the throat when Q cubic feet of water flow through the main per second.

In what units is your answer expressed?

245. A horizontal pipe (area of section 3 square inches) terminates in a standard orifice (area 1 square inch). Three feet from the orifice a piezometer tube indicates 10 feet. How high above the center of the orifice does the free surface of the water in the reservoir supplying the pipe lie?

246. A spherical vessel (radius r) having small orifices (area a) at both top and bottom is full of water. In what time will the vessel empty itself?

247. A hemispherical vessel (radius a) filled with a liquid, specific gravity s, rests with its plane surface on a level table. What force tends to separate the spherical from the plane portion of this vessel?

248. A closed cylindrical vessel (altitude h feet, radius r feet) has a small orifice (area a) in the bottom. Air is supplied to the top of the vessel at a constant pressure of p pounds per square inch gage. Starting with the vessel full of water how many seconds elapse before air escapes from the orifice?

249. A centrifugal pump draws its supply through a vertical pipe 10 inches in diameter. The water leaves the pump through a vertical pipe 6 inches in diameter. A gage attached to the supply pipe shows a pressure of 4 pounds per square inch below atmospheric pressure when the pump delivers 2 cubic feet of water per second. At the same time a pressure gage attached to the delivery pipe 5 feet above the point at which the pressure in the supply pipe is measured reads 40 pounds per square inch. It is found that 42 horse-power is required to drive this pump under the above conditions. Compute its efficiency.

250. How many rectangular weirs 10 feet wide with end contractions must be built in a dam in order to measure a flow of about 6000 cubic feet per minute, if the head on all weirs is to lie between 0.9 and 1.5 feet?

Use Francis' formula and neglect the velocity of approach.

251. A hydraulic turbine is supplied with 12 cubic feet of water per second. The fall is 27 feet. The water enters the turbine through a 12-inch pipe and leaves through a 24-inch pipe which extends below the free surface in the tail-race. Compute the pressure heads in the pipes at (*a*) 12 feet and (*b*) 17 feet below the free surface in the head-race, the turbine lying between these levels.

252. Water flows 4 feet deep with a mean velocity of 2 feet per second in a channel of rectangular section 10 feet wide. It is proposed to build a dam, 6 feet high and with a sharp crest, across this channel. How much will the proposed dam increase the depth of the water in the upper part of the channel? Use Boussinesq's formula and neglect the effect of the velocity of approach.

253. A nozzle has a coefficient of velocity c_v. How much energy is lost per pound of water passing through the nozzle per second in terms of the actual velocity of discharge?

254. Compute the velocity of flow in a fire hose (diameter D, length l, friction factor f) which discharges through a nozzle

(diameter d). The head of water at the inlet to the hose is h feet and the coefficient of velocity of the nozzle is c_v.

255. Develop the fundamental formula for the head lost in pipe friction.

256. A pipe 10 inches in diameter and 3 miles long is supplied with water under a pressure of 1000 pounds per square inch, $f=0.0059$, and the velocity of flow is 5 feet per second. Compute the loss in pressure due to friction.

257. (*a*) Why is it preferable to keep all points of a pipe line below its hydraulic gradient?

(*b*) Under what conditions does a moving fluid exert a force due to its motion?

258. Compute the loss due to the sudden enlargement of a pipe. State clearly upon what experimental basis your computation is based.

259. A straight, horizontal pipe 2000 feet long and 12 inches in diameter contains water moving with a velocity of 5 feet per second. What force does the water exert upon the pipe?

260. Compute the velocity of the water leaving a nozzle attached to a hose, in terms of

d, the diameter of the outlet end of the nozzle;
D, the diameter of the hose;
h, the piezometer reading at the entrance to the nozzle;
c_v, the coefficient of velocity of the nozzle;
and c, the coefficient of discharge of the nozzle.

261. Define: Wetted perimeter, mean hydraulic depth, steady flow.

262. The Chezy formula $v=C\sqrt{ms}$, where m is the mean hydraulic depth, can be derived from the fundamental equation for the loss of head in pipes due to friction. What do the letters C and s represent?

263. Two piezometer tubes placed 100 feet apart in a horizontal pipe of constant diameter d show heights of h_1 and h_2

feet of water ($h_1 > h_2$). What longitudinal force does the flowing water exert on the pipe?

264. What horse-power at the pump is required to maintain a stream of water leaving a nozzle, placed 40 feet above the pump, with a velocity of 80 feet per second? The tip of the nozzle is 1.5 inches in diameter, the coefficient of velocity is 0.90, and the coefficient of contraction is 1.00. The 3-inch hose connecting the pump and the nozzle is 500 feet long and its $f = 0.017$.

265. n lines of hose d feet in diameter and l feet long lead from a fire steamer to a siamese coupling which unites these streams into one carried by a hose d_1 feet in diameter and l_1 feet long to a nozzle whose outlet diameter is d_2 feet. Assuming all hose lines to be horizontal and neglecting the losses due to the siamese coupling and to the nozzle, compute the velocity of the water leaving the nozzle in terms of the friction factors f and f_1, the pressure head h at the steamer, and the given dimensions.

266. Water flows through an eight-inch pipe with a velocity of 5 feet per second under a pressure of 100 pounds per square inch gage, at a point 50 feet above the tail-race. Compute the available power.

267. A turbine discharges into a draught-tube which in turn discharges below the level of the tail-race. The water in the penstock leading to the turbine has the same velocity as the water in the draught-tube. Before leaving the penstock the pressure of the water is 30 pounds per square inch gage. Four feet below the point at which this pressure is read the pressure in the draught-tube is 10 pounds per square inch absolute. How much energy does each pound of water lose in passing through the turbine?

268. A pipe line, 12 inches in diameter, contains a bend (radius 10 feet, angle 90 degrees) lying in a horizontal plane. What force does the water moving with a velocity of 5 feet per second exert upon this bend?

269. What force does water flowing with a velocity of 50 feet per second under atmospheric pressure exert on the pipe shown in Fig. 109?

270. (*a*) Explain with diagram the cause of the reaction exerted by a jet issuing from a standard orifice in the vertical side of a tank.

(*b*) If a represents the area of the standard orifice and h the head of water on its center, compute the reaction of the issuing jet.

271. Upon what quantities does the force exerted by a deflected jet of water depend?

272. Compute the driving force exerted on a Pelton wheel by a jet of water delivering W pounds of water per second with a velocity of v_1 feet per second when the peripheral speed of the center of the buckets is u and the exit angle of the buckets is β.

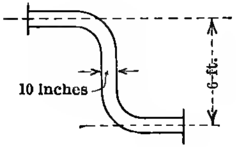

FIG. 109.

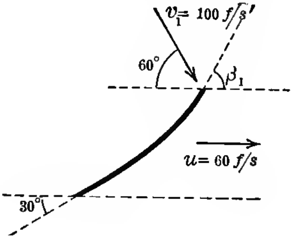

FIG. 110.

273. Compute the velocity of a given bucket on a tangential water wheel in terms of the velocity of the jet so that the power absorbed by the wheel may be a maximum.

274. In Fig. 110, v_1 is the velocity of water entering a vane translating with a velocity u. Compute (*a*) $\tan \beta_1$ if the water is to enter without shock, (*b*) the absolute velocity with which the water leaves the vanes.

275. Develop an expression for the torque exerted on the rotor of a turbine by the water passing through it.

276. A series of vanes, Fig. 111, moving with a velocity u, deflect a jet issuing from a nozzle discharging Q cubic feet of water per second with a velocity of v feet per second. Compute (*a*) the force exerted by the water on the vanes, (*b*) the energy transmitted to the vanes per second, (*c*) the kinetic energy lost by the water per second.

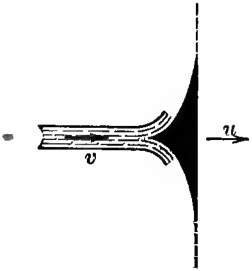

FIG. 111.

277. A jet enters a series of blades, Fig. 112, without shock. The blades move with a velocity u. The initial velocity of the fluid is v_1. Write expressions for the horizontal and the vertical components of the force exerted upon the blades

(*a*) in terms of the relative velocities,

(*b*) in terms of the absolute velocities.

(*c*) Show graphically the equality of the expressions under (*a*) and (*b*).

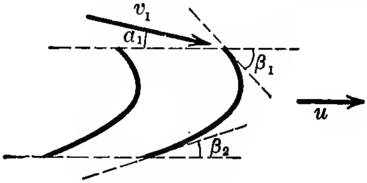

FIG. 112.

(*d*) Write an expression for the work delivered to the blades in terms of the result of (*a*).

(*e*) Show that the loss in kinetic energy of the stream equals the results of (*d*) when friction is neglected.

278. An impulse wheel with radial outward flow operates under a head of 100 feet. If the wheel is designed so that $2\beta_1 = \alpha_1 = 60°$, and the radius at entrance is 2 feet, compute the best speed of the wheel in r.p.m.

279. Express the efficiency of the wheel described in 278 in terms of the radii r_1 and r_2 of the wheel and the angles β_2 and α_1.

280. What relation exists between the magnitudes of the relative velocities at entrance and at exit in

(*a*) an axial-flow impulse wheel,

(*b*) a radial-flow impulse wheel,

(*c*) a turbine,

the effect of friction to be neglected?

281. Prove the relation existing between the relative velocities in a radial-flow impulse wheel.

282. The available head at the nozzle of a Pelton wheel is 300 feet. How many revolutions per minute should the wheel make when running at maximum efficiency?

283. In an outward-flow turbine the water enters without shock and leaves the wheel radially. Assume h, r_1, r_2, A_1, A_2, α_1, and β_2 and compute the fraction of the velocity due to the head h which is developed before the water enters the wheel.

284. The rotor of an inward-flow turbine is fitted with vanes radial at the outer periphery and whose directions make angles of 60° with the radius at the inner periphery. The turbine operates under a head of 40 feet. The radius of the outer periphery of the wheel is 1.5 times the radius of the inner periphery. At what speed must the outer periphery of the wheel move in order that the water may enter it without shock and leave it in a radial direction?

285. An outward-flow turbine is designed to run at 100 r.p.m. under a head of 15 feet. The radii of the rotor are 2 and 3 feet and the corresponding peripheral areas are 5 and 10 square feet. The moving vanes are radial at their inner ends and the water leaves the rotor radially. Compute the velocity with which the water leaves the guide vanes, neglecting frictional losses.

286. Q cubic feet of water are supplied to a radial-flow impulse wheel per second. Compute

(*a*) the torque exerted on the wheel,

(*b*) the energy transmitted to the wheel per second.

Assume all necessary quantities; denote their magnitudes by letters.

287. Water enters an outward-flow turbine without shock and leaves the rotor radially. The vanes are radial at their inner ends. Compute the fraction of the velocity due to the effective head h with which the water enters the rotor. The peripheral areas at entrance and at exit of the rotor are A_1 and A_2 respectively, and the exit angle of the guide vanes is α_1.

288. A turbine is so designed that $\frac{1}{2}$ of the velocity due to the effective head of 10 feet is present at the exit of the guide vanes. Under what pressure in pounds per square inch is the water between the fixed and the moving vanes of this turbine?

289. An outward-flow turbine in which $2\beta_1 = a_1 = 60°$, $a_2 = 90°$, $r_1 = 2$ feet, and $r_2 = 3$ feet is designed to run at 100 r.p.m. How much energy does this turbine extract from each pound of water passing through it?

290. In the design of an outward-flow turbine to run at 180 r.p.m. it is assumed that the radii are 1 and 2 feet, that the crowns are parallel and 6 inches apart, that the terminal angles of the guide blades are 60°. The absolute exit velocity is to be 5 feet per second and radial.

(*a*) Under what effective head can this turbine run?

(*b*) What must be the angles of the moving blades at entrance and at exit?

(*c*) Sketch the outline of a moving blade.

Neglect the thickness of the blades and all frictional resistances.

291. A shaft is to be driven at 150 r.p.m. by means of a Pelton wheel. A supply of 350 cubic feet of water per minute under an effective head of 400 feet is available.

(*a*) What must be the effective radius of the wheel?

(*b*) What horse-power may be expected?

292. An outward-flow impulse wheel has vanes radial at their inner ends and for which $\tan \beta_2 = 60°$, also $r_1 = 4$ and $r_2 = 5$ feet, $\tan \alpha_1 = \frac{4}{3}$ and $v_1 = 200$ feet per second.

(*a*) How many r.p.m. should this wheel make?

(*b*) Compute the velocity of the water leaving the wheel, friction neglected.

293. What is the greatest and the least number of translating vanes 2 feet apart moving 60 feet per second that will be acted upon at any time by a jet of water moving in the direction in which the vanes are moving with a velocity of 100 feet per second?

294. Water flows with a velocity of 20 feet per second and under a pressure of 10 pounds per square inch from a pipe whose sectional area is 4 square inches through a conical pipe to a pipe whose sectional area is 40 square inches. Neglecting friction, find the force exerted by the water on the pipe.

295. Experiment shows that f in $\text{H.P.} = 1.098\, f\, Nn^3(r_2^5 - r_1^5)$ is 0.0013. What should be the diameter of the four plates of a Webb viscous dynamometer if they are wetted to within 6 inches of the center and are to absorb 15 horse-power when running at 600 r.p.m.?

296. Compute the power of the water leaving a nozzle 4 inches in diameter. The pipe line leading to the nozzle is 6000 feet

long and 12 inches in diameter. The free surface of the water in the reservoir lies 1000 feet above the nozzle.

297. Compute the discharge through an 8-inch pipe ($f=0.0056$) 3000 feet long, the outlet being 100 feet below the free surface of the reservoir.

298. A float in the form of a rectangular parallelepiped is to weigh 20 tons with its load. Its length is to be 30 feet and its center of gravity when loaded is to be 5 feet above the bottom.

(*a*) Find the limiting width and draft for stability.

(*b*) Should the draft be made greater or less than the value computed under (*a*) for safety?

299. An opening in a dam of the form shown in Fig. 113 is to be closed by a vertical plate. Assuming this plate to be rigid and weightless where should a single horizontal prop be placed to hold the plate in position?

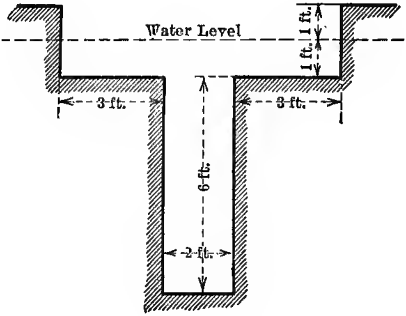

FIG. 113.

300. Illuminating gas fills a vertical main *AB*, Fig. 114. The pressure at *B* is 2 inches of water. Compute the pressure at *A*, 300 feet above *B*, in inches of water.

Assume the specific weights of air and illuminating gas to be constant and equal to 0.08 and 0.05 pound per cubic foot respectively.

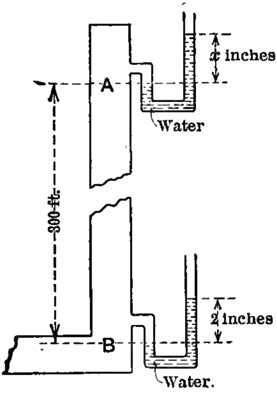

FIG. 114.

ANSWERS

5. 9160 lbs. per sq. in.

6. 5 ins.

7. 20 lbs. per sq. in.

8. 115.3 ft.

9. 14.33 lbs. per sq. in.

10. 14.7 lbs. per sq. in.; 33.9 ft.

12. 26,200 feet = 4.96 miles.

13. 92.3 feet.

15. $p_m = w(sz - h) + p_a$.

17. $p_n - p_m = w[h + z(s-1)]$.

18. (*a*) rises; (*b*) falls.

19. wa^3; $\frac{wa^3}{2}$.

20. $ws\left(c+\frac{h}{2}\right)bh$; $ws\left(c+\frac{2}{3}h\right)\frac{bh}{2}$; $ws\left(c+\frac{h}{3}\right)\frac{bh}{2}$; $ws(c+r)\pi r^2$; $ws\left(c+\frac{2}{3}h \sin\alpha\right)\frac{bh}{2}$.

21. $ws(c+\frac{2}{5}h)(\frac{2}{3}bh)$; $ws(c+\frac{3}{5}h)(\frac{2}{3}bh)$.

22. 0.

23. 0.

24. πwsr^2h.

25. $-\frac{4}{3}ws\pi r^3$; $+\frac{1}{3}ws\pi r^3$; $-\frac{5}{3}ws\pi r^3$.

26. $-ws\pi r^2h$; $+\frac{2}{3}ws\pi r^2h$; $+\frac{1}{3}ws\pi r^2h$.

27. $\frac{\pi r^2 l}{4}ws$.

33. $\frac{2}{3}\left\{\frac{3c^2+3ch+h^2}{2c+h}\right\}$; $\frac{6c^2+8ch+3h^2}{6c+4h}$; $\frac{6c^2+4ch+h^2}{6c+2h}$; $\frac{4c^2+8cr+5r^2}{4(c+r)}$.

34. (3.5 ft., 3 ft.)

35. 6750 lbs.; 3.66 ft.

36. 4150 lbs.; 7650 lbs.

37. 2730 lbs.

39. 187 lbs., 187 lbs.

40. 36.9 lbs.

41. 8540 lbs.; 3.20 ft. below W.L.

42. (*a*) $b' > \frac{wh^2}{2\mu w'h'}$; (*b*) $b' > h\sqrt{\frac{wh}{3w'h'}}$.

43. $b' > h\sqrt{\frac{wh}{3(w'h' - wh)}}$.

46. $w'h'b'^2 + w'h'b'b'' - w'h'b''^2 - wh^3 = 0$.

47. (*a*) 1990 and 264 lbs. per sq.ft.; (*b*) 7.58 ft.

50. 93.6 lbs.

51. $\frac{W_H+W'}{W_H+W}$.

52. $\frac{V-ah}{V-ah'}$.

53. $\frac{w'}{w}l'$.

54. 1.73 ft.

55. $h-h\sqrt[3]{1-\frac{w}{w'}}$.

56. 6.7 ins.

57. $z=\frac{h'}{2}+\frac{b^2}{3h'}$.

58. $b=h\sqrt{6s(1-s)}$; $b=1.2h$.

59. (*a*) Yes, M is 0.0167 above G; (*b*) $>0.789<1$ or $<0.211>0$.

60. $0.384b$ above; $0.0834b$ below; $0.0426b$ above.

61. $>0.281<0.5$.

62. $\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{2r^2}{l^2}}$.

63. When $a<\frac{2k(n-1)}{b}g$, 0. When $a>\frac{2k(n-1)}{b}g$ and $<\frac{nk}{b}g$, $\frac{ab^2l-2gbkl(n-1)}{2g}$. When $a>\frac{nk}{b}g$, $lk\left(b-\frac{n^2kg}{2a}\right)$.

64. $\frac{12kb-b^2}{24k}$.

65. $wh\left(\frac{g-a}{g}\right)$.

66. $\frac{\omega^2r^2}{2g}$.

68. $\omega^2=\frac{4gk(n-1)}{r^2}$.

69. (*a*) 230 r.p.m.; (*b*) 325 r.p.m.

72. $\frac{w}{g}\frac{\omega^2p^2}{2}$.

73. No, No, Yes.

74. 7.5 ft; 2.5 ft.

76. 1.43 sq. ins.

78. $A_2=A_1\sqrt{\frac{\frac{p_a}{w}+h}{h}}$.

79. (*a*) 2.1; (*b*) 2.1 times as large; (*c*) No.

80. $v_2=\sqrt{\frac{2g(h_1+h)}{1-\left(\frac{A_2}{A_1}\right)^2}}$.

82. 29 ft. per sec.

83. 14.3 miles per hour.

85. 41.3 horse-power.

86. 79.5%.

88. $v=\sqrt{\dfrac{gx^2}{2y}}$; $c_0=\sqrt{\dfrac{x^2}{4hy}}$.

91. 0.18%.

92. $0.041\,\dfrac{v_a{}^2}{2g}$.

93. 0.86 cu.ft. per sec.; 105 ft. per sec.

94. $ca\sqrt{2g(h_1-h_2)}$.

95. $\dfrac{2A_1A_2\sqrt{h_1-h_2}}{(A_1+A_2)ca\sqrt{2g}}$.

97. (*a*) $\dfrac{2A}{ca\sqrt{2g}}\left(\sqrt{h_1}-\sqrt{h_2}\right)$; (*b*) $\dfrac{A(h_1-h_2)}{c_ca\dfrac{2A\left(\sqrt{h_1}-\sqrt{h_2}\right)}{ca\sqrt{2g}}}$.

98. $\left(\dfrac{14}{15}\dfrac{\pi}{ca\sqrt{2g}}r^{\frac{5}{2}}=\right)$26.1 minutes.

99. 12.5 minutes.

100. 0.2%.

102. $\dfrac{2cb\sqrt{2g}}{15(h_2-h_1)}\left\{2h_1{}^{\frac{5}{2}}+3h_2{}^{\frac{5}{2}}-5h_1h_2{}^{\frac{3}{2}}\right\}$.

108. 0.412.

109. 0.414.

110. 0.622.

111. 48.0 cu.ft. per sec.

112. 6.1 feet.

115. 48.7 cu.ft. per sec.

117. 23.4 cu.ft. per sec. = 175 gallons per sec.

121. $\dfrac{M}{L^3}$.

124. $C=\sqrt{\dfrac{2g}{f}}$.

125. $d^5-.15d-2=0 \quad \therefore\ d=1.15$ ft. = 14 ins.

126. $1-\dfrac{4flv^2w}{2gpd}$.

127. 99.25%; 364 horse-power.

130. $h=\dfrac{v^2}{2g}\left\{0.5\left(\dfrac{d}{D}\right)^2+f\dfrac{4l}{D}\left(\dfrac{d}{D}\right)^2+\dfrac{1}{c_v{}^2}\right\}$.

131. $d=\sqrt[4]{\dfrac{D^5}{8fl}}$.

132. $\dfrac{h}{3}$.

133. $d=\sqrt[4]{\dfrac{D^5}{4fl}}$.

134. $\dfrac{h}{2}$.

135. (*a*) 6.06 ft. per sec.; (*b*) 15.9 ft. of water (absolute); (*c*) 2.32 cu. ft. per sec.

136. 0.0615 ft.

139. $\dfrac{\pi f w \omega^2 d^5}{80g}$.

141. 0.00142.

142. 0.0013.

143. (*a*) $p_1(A_1 - A_2)$; (*b*) $p_1A_1 - p_2A_2 - \dfrac{A_1v_1w}{g}(v_2 - v_1.)$

144. $\dfrac{Av^2w}{g}\sqrt{2(1-\cos\beta)}$.

145. pA in the direction of arrow v_1.

146. Moment of the couple $= Apa + \dfrac{wv^2A}{g}a$.

147. 423 lbs.

149. 163 lbs. per sq.in.

151. (*a*) $\dfrac{Qwv}{g}$; (*b*) $\dfrac{Qwv}{g}$.

153. $\dfrac{Aw(v_1-u)^2}{2g}\sqrt{2(1+\cos\beta)}$.

154. $\dfrac{Aw(v_1-u)^2}{g}(1+\cos\beta)$.

155. (*a*) $\dfrac{Awv_1}{g}(v_1-u)(1+\cos\beta)$; (*b*) $\dfrac{Awv_1}{g}(v_1-u)u(1+\cos\beta)$.

156. $\sqrt{v_1^2 - 2uv_1 + 2u^2 - 2u(v_1-u)\cos\beta}$.

158. (*a*) $\dfrac{2(v_1-u)(1+\cos\beta)u}{v_1^2}$; (*b*) $u = \dfrac{v_1}{2}$; (*c*) 98.3%, 93.3%.

160. $n = \dfrac{146H}{R^2h^{\frac{3}{2}}}$.

161. (*a*) $n = 11.67$ ∴ 12 jets and 4 wheels; (*b*) 815 r.p.m.; (*c*) 1.165 ins.; (*d*) 1.21 ft.

162. (*a*) 5.51 cu.ft. per sec.; (*b*) 0.041 sq.ft.; (*c*) 2.72 ins.; (*d*) 132 r.p.m.

165. 1. ${}_rv_1^2 = v_1^2 + u^2 - 2v_1u\cos\alpha_1$.

2. $\tan\beta = \dfrac{v_1\sin\alpha_1}{v_1\cos\alpha_1 - u}$.

3. $v_2^2 = 4u^2 + v_1^2 - 4uv_1\cos\alpha_1$.

4. $\tan\alpha_2 = \dfrac{v_1\sin\alpha_1}{2u - v_1\cos\alpha_1}$.

5. $\dfrac{W}{g}2u(v_1\cos\alpha_1 - u)$.

166. (*a*) $u = \dfrac{v_1\cos\alpha_1}{2}$.

(*b*) 1. 90°

2. $2\tan\alpha_1$.

3. $\cos^2\alpha_1$.

4. $v_1\sin\alpha_1$.

167. $\beta_1 = 2\alpha_1 - 90°$; $\alpha_2 = 45° - \frac{\beta_2}{2}$. **170.** 3060 lbs. per ft.

171. (*a*) $m\omega(r_2^2 - r_1^2)$; (*b*) $\frac{m\omega(r_2^2 - r_1^2)}{\Delta T}$; (*c*) $m\omega^2(r_2^2 - r_1^2)$.

172. (*a*) $\omega_2 = \frac{r_1^2}{r_2^2}\omega_1$; (*b*) $m\omega_1^2 r_1^2\left(\frac{r_1^2}{r_2^2} - 1\right)$. **173.** ${}_rv_1^2 - {}_rv_2^2 = u_1^2 - u_2^2$.

174. $\frac{2u_1 \sin \alpha_1}{v_1}$; $\frac{2 \sin \alpha_1 \sin (\alpha_1 - \beta_1)}{\cos \beta_1}$. **175.** α_1.

176.
(*b*) $\beta_1 = 2\alpha_1 - 90°$.
(*c*) $u_1 = \frac{v_1}{2 \sin \alpha_1}$, $\omega = \frac{v_1}{2r_1 \sin \alpha_1}$.
(*d*) $v_2 = \frac{v_1}{\sin \alpha_1}\left(\frac{r_2}{r_1}\right) \sin \left(45° - \frac{\beta_2}{2}\right)$.
(*e*) $\frac{W}{2g}v_1^2\left[1 - \left\{\frac{r_2 \sin \left(45° - \frac{\beta_2}{2}\right)}{r_1 \sin \alpha_1}\right\}^2\right]$.
(*f*) $1 - \left\{\frac{r_2 \sin \left(45° - \frac{\beta_2}{2}\right)}{r_1 \sin \alpha_1}\right\}^2$.

177. $\beta_1 = 34° \, 30'$; $\eta = 94.1\%$. **178.** (*a*) No; (*b*) 517.

179. $v_2 = 0.59v_1$; $\eta = 65.2\%$. **180.** $\frac{W}{g}v_1^2 \sin^2 \alpha_1 \left\{1 - \left(\frac{r_2}{r_1}\right)^2\right\}$

181. ∞.

187.
(*b*) $2u_1^2 = 2gh - v_2^2$.
(*c*) $A_1 v_1 \cos \alpha_1 = A_2 v_2$.
(*d*) $v_1 = K\sqrt{2gh}$, $K = \sqrt{\frac{1}{2 \sin^2 \alpha_1 + \left(\frac{A_1}{A_2}\right)^2 \cos^2 \alpha_1}}$.
(*e*) $u_1 = K \sin \alpha_1 \sqrt{2gh}$.
(*f*) $v_2 = \frac{A_1}{A_2} K \cos \alpha_1 \sqrt{2gh}$.
(*g*) $\eta = 2K^2 \sin^2 \alpha_1$.
(*h*) $\tan \beta_2 = \frac{r_2}{r_1} \frac{A_2}{A_1} \tan \alpha_1$.
(*i*) $Q = A_1 K \cos \alpha_1 \sqrt{2gh}$.
(*j*) $2whA_1 \cos \alpha_1 \sin^2 \alpha_1 \, K^3 \sqrt{2gh}$.

188.

(*a*) $K=\sqrt{\dfrac{1}{2\dfrac{r_1}{r_2}\dfrac{A_1}{A_2}\tan\beta_2\sin\alpha_1\cos\alpha_1+\left(\dfrac{A_1}{A_2}\right)^2\cos^2\alpha_1}}$.

(*b*) $u_1=K\sqrt{2gh}\,\dfrac{A_1}{A_2}\dfrac{r_1}{r_2}\tan\beta_2\cos\alpha_1$.

(*c*) $\eta=\dfrac{u_1K\sqrt{2gh}}{gh}\sin\alpha_1$.

(*d*) $\tan\beta_1=\dfrac{v_1\sin\alpha_1-u_1}{v_1\cos\alpha_1}$.

189.

(*a*) 97.4; 3100.

(*b*) 74° 40′.

(*c*) 5°; 78° 53′.

190. 64.9 ft.

191. 6.35 cu.ft. per sec.

193. $\sqrt{\dfrac{r_2}{r_1}A_1A_2gh\cos\beta_2\cot\alpha_1}$; $2gh\dfrac{r_2}{r_1}\dfrac{A_1}{A_2}\cot\alpha_1\tan\alpha_2$; $1-\dfrac{r_2}{r_1}\dfrac{A_1}{A_2}\cot\alpha_1\tan\alpha_2$.

194. (*a*) 0.225; (*b*) 0.0.

195. $w(b-as)$.

196. 69.3 ft.

198. 38 miles.

200. No.

201. $\frac{1}{3}$ ft.

203. (*a*) πwr^3; (*b*) $\frac{5}{4}r$.

204. 2.58 ins.

205. 65,500 lbs.

206. (*a*) 2.8 ft.; (*b*) 960,000 lbs.

207. Stable; B to $M=\dfrac{\sqrt{2b}}{3}$.

209. $\frac{1}{3}\pi r^2hw$.

210. *a*, to center.

217. 93.0 ft. per sec.

219. 20.9 ft.

220. $1>s>\left(\dfrac{h^2}{h^2+r^2}\right)^3$.

221. $\dfrac{\pi w}{3}r^2h\left(\dfrac{h^2}{h^2+r^2}\right)^3$.

222. $\dfrac{6\pm2\sqrt{3}}{12}$ or 0.789 and 0.212.

223. 44.4 ft. per sec.

225. 4.7 ft.

234. 122 ft. per sec.

235. 31.2 cu.ft. per sec.

236. 8.61 minutes.

238. $\dfrac{\pi r^2}{ac\sqrt{2g}}\left\{2\sqrt{r+h}-\dfrac{16}{15}\sqrt{r}\right\}$.

239. 6.75 ft.

240. 4.78 cu.ft. per sec.

242. 5.42 ft. per sec.; 7.95 gal. per sec.

243. 0.425 ft. per sec.

244. $\frac{4wQ^2}{gA^2}$ lbs. per sq. ft.

245. 10.4 ft.

246. $\frac{16}{15}\frac{\pi}{ca\sqrt{2g}}r^2\sqrt{2r}$.

247. $\frac{1}{3}\pi r^3 ws$.

248. $\frac{2\pi r^2}{ac\sqrt{2g}}\left\{\sqrt{2.31p+h}-\sqrt{2.31p}\right\}$.

249. 58.3%.

250. 3.

251. (*a*) 8.31 ft.; (*b*) −10.23 ft.

252. 3.77 ft.

253. $\frac{v^2}{2g}\left(\frac{1}{c_v^2}-1\right)$.

254. $v^2=\dfrac{2gh}{\dfrac{4fl}{D}+\dfrac{1}{c_v^2}\left(\dfrac{D}{d}\right)^4-1}$.

255. $h_f=f\frac{4}{d}l\frac{v^2}{2g}$.

256. 75.5 lbs. per sq.in.

259. 764 lbs.

260. $v=c_v\sqrt{\dfrac{2gh}{1-\left(\dfrac{d}{D}\right)^4c^2}}$.

263. $\frac{w\pi d^2}{4}(h_1-h_2)$.

264. 112.5 horse-power.

265. $\sqrt{\dfrac{2gh}{1+\left(\dfrac{d_2}{d_1}\right)^4\dfrac{4f_1l}{d_1}+\dfrac{1}{n^2}\left(\dfrac{d_2}{d_1}\right)^4\dfrac{4fl}{d}-\dfrac{1}{n^2}\left(\dfrac{d_2}{d_1}\right)^4}}$.

266. 55.7 horse-power.

267. 84 ft.-lbs.

268. 54.4 lbs.

269. Moment of couple = 15,900 lb.-ft.

272. $\frac{W}{g}(v_1-u)(1-\cos\beta)$.

274. (*a*) 8.67; (*b*) 46.15 ft. per sec.

276. (*a*) $\frac{Qw}{g}(v-u)$; (*b*) $\frac{Qw}{g}(v-u)u$; (*c*) $\frac{Qw}{g}(v-u)u$.

277. (*a*) $\frac{W}{g}({}_rv_1\cos\beta_1+{}_rv_2\cos\beta_2)$; (*b*) $\frac{W}{g}({}_rv_1\sin\alpha_1-{}_rv_2\sin\alpha_2)$.

278. 220 r.p.m.

279. $1-\left\{\dfrac{r_2\cos\left(45-\dfrac{\beta_2}{2}\right)}{r_1\sin\alpha_1}\right\}$.

282. $\dfrac{658}{r}$.

283. $$\sqrt{\frac{1}{2\frac{r_1}{r_2}\frac{A_1}{A_2}\tan\beta_2\sin\alpha_1\cos\alpha_1+\left(\frac{A_1}{A_2}\right)^2\cos^2\alpha_1}}.$$

284. 34.7 ft. per sec. **285.** 27.8 ft. per sec.

287. $$\sqrt{\frac{1}{2\sin^2\alpha_1+\left(\frac{A_1}{A_2}\right)^2\cos^2\alpha_1}}.$$

288. 3.26 lbs. per sq.in., gage. **289.** 20.5 ft.-lbs.

290. (*a*) 10.54 ft.; (*b*) $\beta_1=8° 42'$, $\beta_2=82° 60'$.

291. (*a*) 5.09 ft.; (*b*) 212.

292. (*a*) 382 r.p.m.; (*b*) 42.4 ft. per sec.

293. 3 and 2. **294.** 447 lbs.

295. 29.2 ins. **296.** 642 horse-power.

297. 2.77 cu.ft. per sec.

298. (*a*) draft, 212; width, 10.04 ft.; (*b*) less.

299. 5.44 ft. below the top of the dam.

300. 3.73 ins. of water.

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